

GROUNDWORK FOR A MATHEMATICAL PERSPECTIVISM

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- At some level, every mathematician is a pluralist.
- The “correct” theory is the one that will yield a proof of an especially recalcitrant theorem or deliver understanding.
- There are deeper philosophical questions about pluralism. Is a particular theory true? Is there even a unique true theory?
- But generally speaking, if a mathematician can adopt a different perspective for some end, then they’ll take it.

- This hasn't stopped a debate going on between advocates of **set-theoretic** and **category-theoretic** foundations.
- Each accuses the other side of committing various philosophical and mathematical **sins** (e.g. [Mac Lane, 1986], [Hellman, 2006]).
- Others (e.g. [Linnebo and Pettigrew, 2011], [Barton and Friedman, 2019]) suggest a more **pluralistic** outlook, on which each should be understood as about a different **subject matter**.

MAIN AIM.

Argue for a new(ish) kind of position in the philosophy of
mathematics: **Perspectivism**.

INTRODUCTION

SET THEORY AND CATEGORY THEORY

PLURALISM AND PERSPECTIVISM: THE CASE OF
CLASSICAL MECHANICS

HOW PERSPECTIVAL IS THE DIFFERENCE BETWEEN SET
AND CATEGORY THEORY?

CONCLUSIONS AND OPEN QUESTIONS

- We'll be assessing theories qua **foundations**.
- For me, a foundation comprises:
 - (1.) A *formal language*.
 - (2.) A family of *axiomatic theories* in that language.
 - (3.) A rough-and-ready description of what that language is *about*; how the language should be *understood* (note this is a *metasemantic* claim).

- There's various **desiderata** we'd like from a foundation (see Penelope Maddy, e.g. [Maddy, 2019]).
- The ones relevant for us:

- **Generous Arena.** Find *representatives* for our usual mathematical structures (e.g. the natural numbers, the real numbers) using our foundational theory.
- **Metamathematical Corral.** Provide a theory in which metamathematical investigations of **relative provability** and **consistency strengths** can be conducted.
- **Risk Assessment.** Provide a **degree of confidence** in theories commensurate with their consistency strength.
- **Essential Guidance.** “...capture the fundamental character of mathematics as it’s actually done, that will guide mathematicians toward the truly **important concepts** and **structures**, without getting bogged down in irrelevant details.” [Maddy, 2017, p. 305]

- **Set theory:** Single primitive $\in$, various axiomatisations trying to get at the **universe(s)** of sets.
- **Category theory:** Idea of arrow, domain and co-domain, good for systematising algebraic notion of what it takes to be a **map** and isolating notions of **structure/information preservation**.

- TLDR version:
- Set theory is **great** for **Generous Arena**, **Metamathematical Corral**, and **Risk Assessment**, but is **bad** for **Essential Guidance**.
- Category theory is **great** for **Essential Guidance**, can do **Metamathematical Corral**, and **Generous Arena**, gives us little **Risk Assessment**.

CLAIM.

They are **dual** because of what they are **about**.

- Set theory is about building up **concreta**, category theory is about representing **schematic relationships** (see [Landry and Marquis, 2005]).
- So: **Obviously** we should just be pluralists — they are **about different things** (cf. [Barton and Friedman, 2019]).

- I'll set up what I mean by the distinctions:

PLURALISM.

Pluralism about a given field F is the position that there are multiple legitimate theories concerning F (in our case these will be foundational frameworks).

STRONG PLURALISM.

A view is strongly pluralist if, in addition to pluralism, it holds that there is little *shared* between the (natural interpretations of) the two theories.

PERSPECTIVISM.

We should be pluralists, but only insofar as our frameworks provide **alternative perspectives** on the same subject matter.

CLAIM.

The strength of pluralism is **graded** (not pluralism/monism).

CLAIM.

As more is shared between two frameworks, the differences become more **perspectival**.

- **Example** of **strong** pluralism: Nancy Cartwright's "dappled world" (e.g. Hooke's Law in beam mechanics).
- **Example** for **perspectivism**: Giere's example of dichromates vs. trichromates.
- **Example** for **perspectivism**: The 2Ders looking at a 3D-object.
- **Example** of **strongish** pluralism: Difference between Quantum Mechanics and General Relativity.

- **Example** for perspectivism (very **weak** pluralism):
Hamiltonian and Lagrangian formulations of classical mechanics.
- In addition to position, time, and mass you also have:
 1. **Hamiltonian:** Momenta
 2. **Lagrangian:** Velocities
- These are **roughly** equivalent: For any reasonable experiment you could get the **same prediction** from each.

- The situation with **equivalence** is surprisingly **subtle**.
- For example, the **Legendre transform** let's you move (modulo enabling assumptions) between Lagrangian and Hamiltonian.
- But whether or not these turn out to be equivalent **depends** on:
 1. How the theories are **fomulated**.
 2. What notion of **equivalence** we pick.
- **Example.** [Barrett, 2019] Whether the categories of models for Hamiltonian and Lagrangian mechanics are equivalent **depends** on how you formulate the theories.

- Two things we should want to increase our credence that there is **shared content**.

TRANSLATION.

- In order for the difference between two frameworks (or portions of frameworks) to be considered *perspectival*, there should be ways of **moving** between the relevant formal theories (e.g. via **interpretations**).
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- An **increase** in “similarity” between the different perspectives (e.g. witnessed by an increase in the robustness of the interpretability phenomenon, up to logical equivalence), indicates a **decrease** in the strength of pluralism being offered.

INTERACTION EFFECTS.

The existence of **interaction effects** is evidence that the difference between two frameworks is **more perspectival** and less strongly pluralist.

- What do I mean by **interaction effects**?
- e.g. The Legendre transform (often) provides one for Hamiltonian and Lagrangian mechanics.
- But it might be that there's **no known general transform** as when we have somewhat **overlapping regimes** (e.g. Hawking radiation).
- This can be very **minimal**! But the more interaction effects, the less strongly pluralist, the more perspectival for the overlapping regimes.

So:

QUESTION.

To what extent is the difference between set-theoretic and category-theoretic foundations **perspectival**?

- We'll consider the following candidate **category** theories:
 - CCAF (the *Category of Categories as an Autonomous Foundation*) has the following axioms:¹
 - Existence of finite categories **1**, **2**, and **3**.
 - Arrow extensionality.
 - The existence of products and coproducts for any two categories.
 - Every parallel pair of arrows (functors) has an equalizer and a co-equalizer.
 - For any two categories, the functor category between them exists.
 - There is a natural number category.
 - Categorical choice.

¹This is adapted from Chapter 9 of [McLarty, 2008].

- ETCS^+ contains the axioms of a **well-pointed topos** with a **natural number object** and satisfying (i) a categorical version of the **Axiom of Choice** (epics split), and (ii) a categorical version of the **Axiom of Replacement**.
- CCAF^+ is CCAF plus the axiom that **there is** a category **Set** satisfying ETCS^+ , plus the **Fullness principle** that: For all sets $I, J : \mathbf{1} \rightarrow \mathbf{Set}$, every functor $(1 \downarrow I) \rightarrow (1 \downarrow I)$ equals $(1 \downarrow I)$ for some function $f : I \rightarrow J$ in **Set**.

- We'll consider the following **set** theories:
- ZFC
- (That's all... apart from the odd large cardinal for **flavour**.)

- Two theories are **bi**-interpretable (a pretty robust kind of theoretical similarity), when you can define translations each way, such that composing translations gets you something isomorphic with what you started with.
- A tweak:

DEFINITION.

[Meadows, F] Two theories T_1 and T_2 are χ -*interpretable* when there are interpretations $F : \mathsf{T}_1 \rightarrow \mathsf{T}_2$ and $G : \mathsf{T}_2 \rightarrow \mathsf{T}_1$, such that:

- (I) If $M \models \mathsf{T}_1$, then $G \circ F(M)$ is **categorially equivalent** to M
- (II) IF $N \models \mathsf{T}_2$, then $F \circ G(N)$ is **categorially equivalent** to N .

THEOREM

[Mitchell, 1972] [Osius, 1974] (see [Meadows, F] for this presentation) There are interpretations $\text{catset} : \mathcal{L}_{\text{cat}} \rightarrow \mathcal{L}_{\in}$ and $\text{setcat} : \mathcal{L}_{\in} \rightarrow \mathcal{L}_{\text{cat}}$ such that:

- (1.) If $M \models \text{ZFC}$, then $\text{catset} \circ \text{setcat}(M)$ is *categoryally equivalent* to M .
- (2.) If $N \models \text{ETCS}^+$ then $\text{setcat} \circ \text{catset}$ is *categoryally equivalent* to N .

FACT

(Meadows, personal communication.) ETCS^+ and ZFC are *not* bi-interpretable.^a

^aThe result is known, but the subject of current work by Meadows, and so I won't include a description of the proof here.

INTERACTION EFFECTS

- Isbell's result that $\mathbf{Set}^{Op}$ is **bounded** iff there's **no** proper class of measurable cardinals.
- Prague school in the 1960s: Existence of rigid classes in categories such as graphs, or metric spaces or compact Hausdorff spaces with continuous maps, **equivalent** to Vopěnka's Principle.
- Another (*relatively* simple) example: The category **Ord** of ordinals cannot be fully embedded into the category **Gra** of graphs.
- Bagaria and Brooke-Taylor's result on colimits and elementary embeddings [Bagaria and Brooke-Taylor, 2013].
- These are very **direct** interaction effects, **assuming** the sets instantiate the relevant category-theoretic structure.

- So it looks like ETCS^+ and ZFC are **different** but **somewhat** perspectival.
- But what about CCAF^+ ?
- TLDR version: There's some real **challenges**, but it's open(ish).
- For **sure** the Osius-Mitchell translation is problematic, basically because of **functor categories**.
- ZFC and CCAF^+ for sure **aren't** bi-interpretable (or anything that requires isomorphism), but it's at least possible that there's **extensions** that are χ -interpretable or stronger.

CONCLUSIONS

- Some evidence that there's **shared content** and **target** for category-theoretic and set-theoretic foundations.
- **Note:** This is **good** when we start to consider $\mathbb{N}$ in each!
- But it can become **greater** when we start to take on additional commitments / theories (e.g. ETCS^+ vs. ETCS^+ **within** CCAF^+ and ZFC).
- But this is just the **start** (**groundwork!**)

Thanks! Discussion!

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