

GROUNDWORK FOR A MATHEMATICAL PERSPECTIVISM

Neil Barton

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<https://neilbarton.net/blog/>



NUS
National University
of Singapore

- At some level, every mathematician is a pluralist.
- The “correct” theory is the one that will yield a proof of an especially recalcitrant theorem or deliver understanding.
- There are deeper philosophical questions about pluralism. Is a particular theory true? Is there even a unique true theory?
- But generally speaking, if a mathematician can adopt a different perspective for some end, then they’ll take it.

- This hasn't stopped a debate going on between advocates of **set-theoretic** and **category-theoretic** foundations.
- Each accuses the other side of committing various philosophical and mathematical **sins** (e.g. [Mac Lane, 1986], [Hellman, 2006]).
- Others (e.g. [Linnebo and Pettigrew, 2011], [Barton and Friedman, 2019]) suggest a more **pluralistic** outlook, on which each should be understood as about a different **subject matter**.

MAIN AIM.

Argue for a new(ish) kind of position in the philosophy of
mathematics: **Perspectivism**.

INTRODUCTION

SET THEORY AND CATEGORY THEORY

PLURALISM AND PERSPECTIVISM: AN INTERLUDE FROM
THE PHILOSOPHY OF SCIENCE

HOW PERSPECTIVAL IS THE DIFFERENCE BETWEEN SET
AND CATEGORY THEORY?

CONCLUSIONS AND OPEN QUESTIONS

- We'll be assessing theories qua **foundations**.
- For me, a foundation comprises:
 - (1.) A *formal language*.
 - (2.) A (family of) *axiomatic theory (or theories)* in that language.
 - (3.) A rough-and-ready description of what that language is *about*; how the language should be *understood* (note this is a **metasemantic** claim).

- There's various **desiderata** we'd like from a foundation (see Penelope Maddy, e.g. [Maddy, 2019]).
- The ones relevant for us:

- **Generous Arena.** Find *representatives* for our usual mathematical structures (e.g. the natural numbers, the real numbers) using our foundational theory.
- **Metamathematical Corral.** Provide a theory in which metamathematical investigations of relative provability and consistency strengths can be conducted.
- **Risk Assessment.** Provide a degree of confidence in theories commensurate with their consistency strength.
- **Essential Guidance.** “...capture the fundamental character of mathematics as it’s actually done, that will guide mathematicians toward the truly important concepts and structures, without getting bogged down in irrelevant details.” [Maddy, 2017, p. 305]

- **Set theory:** Single primitive $\in$, various axiomatisations trying to get at the **universe(s)** of sets.
- **Category theory:** Idea of arrow, domain and co-domain, good for systematising algebraic notion of what it takes to be a **map** and isolating notions of **structure/information preservation**.

- TLDR version:
- Set theory is **great** for **Generous Arena**, **Metamathematical Corral**, and **Risk Assessment**, but is **bad** for **Essential Guidance**.
- Category theory is **great** for **Essential Guidance**, can do **Metamathematical Corral**, and **Generous Arena**, gives us no **Risk Assessment**.
- **Claim:** They are **dual** because of what they are **about**.
- Set theory is about building up **concreta**, category theory is about representing **schematic relationships** (see [Landry and Marquis, 2005]).
- So: **Obviously** we should just be pluralists — they are **about different things** (cf. [Barton and Friedman, 2019]).

- I'll set up what I mean by the distinctions:

PLURALISM.

Pluralism about a given field F is the position that there are **multiple legitimate** theories concerning F (in our case these will be foundational frameworks).

STRONG PLURALISM.

A view is **strongly pluralist** if, in addition to pluralism, it holds that there is significant **incommensurability** between the (natural interpretations of) the two theories.

PERSPECTIVISM.

We should be pluralists, but only insofar as our frameworks provide **alternative perspectives** on the same subject matter.

CLAIM.

The strength of pluralism is **graded** (not pluralism/monism).

- **Example** of **strong** pluralism: Nancy Cartwright's "dappled world" (e.g. Hooke's Law in beam mechanics).
- **Example** for **perspectivism**: Giere's example of dichromates vs. trichromates.
- **Example** for **perspectivism**: The 2Ders looking at a 3D-object.
- **Example** of **strongish** pluralism: Difference between Quantum Mechanics and General Relativity (scales, probabilistic).

- **Example** for perspectivism (very **weak** pluralism):
Newtonian, Hamiltonian, and Lagrangian formulations of classical mechanics.
- In addition to position, time, and mass you also have:
 1. **Newtonian:** Forces
 2. **Hamiltonian:** Momenta
 3. **Lagrangian:** Velocities
- These are **roughly** equivalent: For any reasonable experiment you could get the same prediction from each (with varying degrees of effort).

- The situation with **equivalence** is surprisingly **subtle**.
- For example, the **Legendre transform** let's you move (almost always) between Hamiltonian and Lagrangian.
- But whether or not these turn out to be equivalent **depends** on:
 1. How the theories are **fomulated**.
 2. What notion of **equivalence** we pick.
- **Example.** [Barrett, 2019] Whether the categories of models for Hamiltonian and Lagrangian mechanics are equivalent **depends** on how you formulate the theories.

- **Suggestion.** The **strength** of the pluralism involved is a matter of **degree**.
- It's not a case of being pluralist or not, it's rather a case of **how** different the target looks and how much **shared content** is there?
- There's a big question here about how **metasemantics** feeds into the picture, and how this affects meaning, but I'll suppress this!

- Two things we should want to convince ourselves that there is **shared content**.

TRANSLATION.

In order for the difference between two frameworks to be considered *perspectival*, there should be **interpretability** between the relevant formal theories. An **increase** in “similarity” between the different perspectives (e.g. witnessed by an increase in the robustness of the interpretability phenomenon, up to logical equivalence), indicates a **decrease** in the strength of pluralism being offered.

- Lagrangian and Hamiltonian versions of classical mechanics **partially** satisfy this criterion.

INTERACTION EFFECTS.

The existence of **interaction effects** is evidence that the difference between two frameworks is **more perspectival** and less strongly pluralist.

- What do I mean by **interaction effects**?
- e.g. The Legendre transform provides one for Hamiltonian and Lagrangian mechanics.
- But it might be that there's **no known general transform**.
e.g. Quantum Mechanics, General Relativity, and Hawking radiation.

So:

QUESTION.

To what extent is the difference between set-theoretic and category-theoretic foundations **perspectival**?

- We'll consider the following candidate **category** theories:
 - CCAF (the *Category of Categories as an Autonomous Foundation*) has the following axioms:¹
 - Existence of finite categories **1**, **2**, and **3**.
 - Arrow extensionality.
 - The existence of products and coproducts for any two categories.
 - Every parallel pair of arrows (functors) has an equalizer and a co-equalizer.
 - For any two categories, the functor category between them exists.
 - There is a natural number category.
 - Categorical choice.

¹This is adapted from Chapter 9 of [McLarty, 2008].

- $\mathbf{ETCS}^+$ contains the axioms of a well-pointed topos with a natural number object and satisfying (i) a categorical version of the Axiom of Choice (epics split), and (ii) a categorical version of the axiom of replacement.
- $\mathbf{CCAF}^+$ is $\mathbf{CCAF}$ plus the axiom that there is a category satisfying $\mathbf{ETCS}^+$, plus the Fullness principle that: For all sets $I, J : \mathbf{1} \rightarrow \mathbf{Set}$, every functor $(1 \downarrow I) \rightarrow (1 \downarrow I)$ equals $(1 \downarrow I)$ for some function $f : I \rightarrow J$ in $\mathbf{Set}$.

- We'll consider the following **set** theories:
- ZFC
- (That's all... apart from the odd large cardinal for **flavour**.)

- Two theories are **bi**-interpretable (a pretty robust kind of theoretical similarity), when you can define translations each way, such that composing translations gets you something isomorphic with what you started with.
- A tweak:

DEFINITION.

[Meadows, F] Two theories T_1 and T_2 are χ -*interpretable* when there are interpretations $F : \mathsf{T}_1 \rightarrow \mathsf{T}_2$ and $G : \mathsf{T}_2 \rightarrow \mathsf{T}_1$, such that:

- (I) If $M \models \mathsf{T}_1$, then $G \circ F(M) \equiv_{cat} M$
- (II) If $N \models \mathsf{T}_2$, then $F \circ G(N) \equiv_{cat} N$.

THEOREM

[Mitchell, 1972] [Osius, 1974] (see [Meadows, F] for this presentation) There are interpretations $\text{catset} : \mathcal{L}_{\text{cat}} \rightarrow \mathcal{L}_{\in}$ and $\text{setcat} : \mathcal{L}_{\in} \rightarrow \mathcal{L}_{\text{cat}}$ such that:

- (1.) If $M \models \text{ZFC}$, then $\text{catset} \circ \text{setcat}(M)$ is categorially equivalent to M .
- (2.) If $N \models \text{ETCS}^+$ then $\text{setcat} \circ \text{catset}$ is categorially equivalent to N .

FACT

(Meadows, personal communication.) ETCS^+ and ZFC are *not* bi-interpretable.^a

^aThe result is known, but the subject of current work by Meadows, and so I won't include a description of the proof here.

INTERACTION EFFECTS

- Isbell's result that $\mathbf{Set}^{Op}$ is bounded iff there's no proper class of measurable cardinals.
- Prague school in the 1960s: Existence of rigid classes in categories such as graphs, or metric spaces or compact Hausdorff spaces with continuous maps, equivalent to Vopěnka's Principle.
- Another (*relatively* simple) example: The category **Ord** of ordinals cannot be fully embedded into the category **Gra** of graphs.
- Bagaria and Brooke-Taylor's result on colimits and elementary embeddings [Bagaria and Brooke-Taylor, 2013].
- These are very **direct** interaction effects, assuming the category theory applies to the sets.

- So it looks like ETCS^+ and ZFC are **different** but **somewhat** perspectival.
- But what about CCAF^+ ?
- TLDR version: There's some real **challenges**, but it's open.
- For **sure** the Osius-Mitchell translation is problematic, basically because of functor categories.
- ZFC and CCAF^+ for sure **aren't** bi-interpretable (or anything that requires isomorphism), but it's at least possible that there's **extensions** that are χ -interpretable or stronger.

CONCLUSIONS

- Some evidence that there's **shared content** and **target** for category-theoretic and set-theoretic foundations.
- **Note:** This is **good** when we start to consider $\mathbb{N}$ in each!
- But it can become **greater** when we start to take on additional commitments / theories (e.g. ETCS^+ vs. ETCS^+ **within** CCAF^+ and ZFC).
- But this is just the **start** (**groundwork!**)
- **Time** check, possible thanks!

- What do we think of the notion of semantic preservation (and degrees thereof) at play?
- e.g. as witnessed by formal theories.

- What about interaction effects?
- Are they indicative of being in a **perspectival** situation?

- **How should we handle inconsistent information?**
- e.g. via Brown and Priest's chunk-and-permeate?
- Paraconsistent logic?
- Tools from the perspectivism literature (e.g. in Michaela Massimi's work)?

- Can we formalise the notion of perspective to good effect?

SUMMARY

1. What do we think of the notion of semantic preservation (and degrees thereof) at play?
2. What about interaction effects?
3. How should we handle inconsistent information?
4. Can the notion of a perspective be formalised to good effect? (And obvious metaperspectivism worries.)

Thanks! Discussion!

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