

WHAT DOES IT MEAN TO “JUSTIFY” A SET THEORY?

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- We've been interested in the idea of set-theoretic **justification** for a **long time**.
- From Cantor on powerset, to Zermelo on well-ordering / Choice, to contemporary discussions of the justification of various determinacy hypotheses, the Continuum Hypothesis, and more.
- After the independence of CH (among many other questions) set theory has roughly **split**...

MONISM.

Monists (about some question Q) believe that Q is **determinate** (either true or false) and that the determinacy of Q is **necessary** to engage in **justification** regarding Q .

PLURALISM.

Pluralists (about some question Q) believe that Q is **indeterminate** (neither true nor false) and that **justification** of Q is **impossible**.

- Monists and pluralists **agree** that **determinacy of truth** is **necessary** for **justification**, but they **disagree** whether Q is so determinate.

TARGET.

Argue that the monist and the pluralist are **missing possibilities**.

INTRODUCTION

THE TWO EXTREMES: MONISM AND
MULTIVERSISM

CHOICELIKE VS. TRUTHLIKE JUSTIFICATION

METASEMANTICS AND OBJECTIVITY

CONCLUSION

- Let's examine the two **extreme** positions, starting with Gödel.

*It is to be noted, however, that even if one should succeed in proving its [CH] undemonstrability as well, this would ... by no means settle the question definitively. Only someone who ... denies that the concepts and axioms of classical set theory have any meaning (or any well-defined meaning) could be satisfied with such a solution, not someone who believes them to describe some well-determined reality. For in this reality Cantor's conjecture must be either **true** or **false**...[Gödel, 1947, pp. 519–520]*

*...and its undecidability from the axioms as known today can only mean that these axioms **do not contain a complete description** of this reality; and such a belief is by no means chimerical, since it is **possible** to point out ways in which a **decision of the question**, even if it is undecidable from the axioms in their present form, might nevertheless be obtained. [Gödel, 1947, pp. 519–520]*

- And later, his oft-discussed conception of **intrinsic** and **extrinsic** justification.

*For first of all the axioms of set theory by no means form a system closed in itself, but, quite on the contrary, the **very concept** of set on which they are based suggests their extension by new axioms which assert the existence of still further iterations of the operation “set of”...But probably there exist others based on **hitherto unknown principles**... which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts. [Gödel, 1947, pp. 520–521]*

*Furthermore, however, even **disregarding** the intrinsic necessity of some new axiom, and even in case it had no intrinsic necessity at all, a decision about its truth is possible also in another way, namely, inductively by studying its “success”, that is, its **fruitfulness in consequences** and **in particular** in “verifiable” consequences, i.e., **consequences demonstrable without the new axiom** [Gödel, 1947, p. 521]*

The picture:

1. We have a background **ontological** realism.
2. We think we have **unproblematic** (though not necessarily **easy**) access to that reality.
3. We have **sufficiently good** access that we can **justify** the **truth** of new axioms (either **conceptually true** or justified via **prediction** and **confirmation**).

Since then the Gödelian idea of **fruitfulness** has been **broadened** to include other desiderata e.g. [Maddy, 1988a], [Maddy, 1988b], [Maddy, 2011]

It has become customary to describe these two rough categories of justification as “intrinsic” — self-evident, intuitive, part of the “concept of set”, and such like — and “extrinsic” — effective, fruitful, productive. [Maddy, 2011, p. 47]

- But **why** should these be true?

*The central problem in the philosophy of natural science is when and **why** the sorts of facts scientists cite as evidence **really are** evidence. The same is true in the case of mathematics. [Maddy, 1988a, p. 481]*

PUZZLE.

Why should we think that these “abductive” reasons track the **truth**?

- Pluralists by contrast think that the project of justification is **hopeless** (at least for certain questions):

*... for any particular attempt at a dream solution of the continuum problem, where a new set-theoretic principle is proposed and proved to settle CH, then I predict that set-theorists will object to the claim that the principle is a **set-theoretic truth**, and furthermore, their objections will arise from a wellspring of deep mathematical experience with the contrary hypothesis. [Hamkins, 2015, p. 140]*

*On the multiverse view, consequently, the continuum hypothesis is a **settled question**; it is incorrect to describe the CH as an open problem. The answer to CH consists of the expansive, detailed knowledge set theorists have gained about the extent to which it holds and fails in the multiverse... [Hamkins, 2012, p. 430]*

The picture:

1. There's a background of **indeterminacy** (possibly underwritten by multiple universes or vagueness).
2. This means that the project of justification is **futile**, how could you justify that which is **indeterminate**?

Summed up by Koellner:

*The pluralist may grant that the results are quite interesting and that they show that $\text{AD}^{L(\mathbb{R})}$ has a certain **appeal** since, for example, it renders $L(\mathbb{R})$ a **paradise** for analysts, but they will view this as a **practical reason** for adopting $\text{AD}^{L(\mathbb{R})}$, not a **theoretical reason**. On this view there are many roads that one can take — $V = L$, $\text{AD}^{L(\mathbb{R})}$, etc. and that while each has its **advantages** with respect to certain aims there is no “**fact of the matter**” as to which is the correct one, any more than there is a “fact of the matter” as to which of two notations is the correct notation. [Koellner, 2013, §5]*

- There's a **crass** way of interpreting this debate (which, to be **clear**, I do **not** think Koellner, Maddy, or Hamkins are committing to):

THE CRASS INTERPRETATION.

This is all a matter of the amount of mathematical **determinacy/indeterminacy** and our **access** to it.

1. Determinacy is **necessary** for justification.
2. Access to that truth/falsity is **necessary** for justification.

- But something **subtle** is going on.
- Even really vociferous pluralists do **not** think that set theory is an **anything goes** affair:

*The multiverse view is one of higher-order realism — Platonism about universes — and I defend it as a realist position asserting actual existence of the alternative set-theoretic universes into which our mathematical tools have allowed us to glimpse. The multiverse view, therefore, does not reduce via proof to a brand of formalism. In particular, we may **prefer** some of the universes in the multiverse to others, and there is **no** obligation to consider them **all** as somehow **equal**. [Hamkins, 2012, p. 417]*

Now for something **completely** different:

- Picking up on work by Bob Beddor: There's a distinction between what we might call **truth-directed justification** and **choice-directed justification** .
- Consider...

Pat the perceiver is having an experience as of a red vase.
They might be in:

1. **Good case.** They're looking at a red vase.
 2. **Bad case.** They're looking at a cleverly lit white vase.
 3. **Really bad case.** Their brain is behaving suitably badly, and they're completely hallucinating a red vase.
- **Claim.** The **Good Case** and **Bad Case** share some sort of **normative virtue** that the **Really Bad Case** lacks, however **blameless** everyone involved might be.

Murali the moral agent wants to perform an action:

1. **Good case.** They research a charity, donate, and help.
 2. **Bad case.** They research a “charity” and donate. But unbeknownst to them, they have been duped by a cleverly disguised webpage, and end up donating to a terrorist organisation.
 3. **Really bad case.** Their brain is behaving suitably badly, and they believe that donating to the terrorist organisation is the right thing to do.
- **Claim.** The **Good Case** and **Bad Case** share some sort of **normative virtue** that the **Really Bad Case** lacks, however **blameless** everyone involved might be.

- We can both justify that a belief is true and justify a choice.
- There is a positive normative valence to each.
- Adding to our stock of accepted axioms is a choice.
- The mistake that the crass interpretation makes is to assume that this choice has to be underwritten by truth and access.

- You can be truth-value realist but accept that the only relevant kind of justification is **choicelike**.
- You can be a truth-value anti-realist but accept that there is a **unique** normatively best choice we can make.
- e.g. Suppose there's a unique **best** set theory for doing physics.

- There is something **right** about the crass interpretation, however.
- Assuming that you do want to accept the truths of set theory — that non-defeated truthlike justification is **sufficient** for (or perhaps strongly supports) the justification of accepting an axiom, then there is a **connection** between determinacy, access, and justification.
- So we want to know: Where do our truthlike justifications **end** and our merely choicelike justifications **begin**?

- Part of the issue here, I contend, concerns **metasemantics**.
- The standard attitude (partly underlying the crass view).
- You have some level of **expressive resources** concerning the sets, and this **circumscribes** the **range of determinacy** and the **possible** range of your **truthlike justifications**.
- The only question to be answered is your **epistemic access** to the range of truthlike justifications.

- But this account commits us to certain principles of **metasemantics**.
- In particular, that increasing your commitments (perhaps on the basis of choicelike justifications) doesn't change your possibilities for **truthlike** justifications.

- How much determinacy you get is a function of:
 1. **Ontology.** What **exists**?
 2. **Metasemantics.** How do words get their **meaning**?
 3. **Correctness.** Once words have their meaning, how **truth/correctness** is determined.
- So, for instance, you can think that no sets exist, but there's a lot of correct claims about mathematics because you think it should be interpreted as about possibility (e.g. if you're an in re structuralist like Geoffrey Hellman).

- **Example. Strong Lewisian Reference**
Magnetism. There are entities that are more and less **eligible** to be the referents of terms, and there is a most eligible referent for our discourse involving sets.
- **Example. Soysal-style Descriptivism.** The meaning of our set-theoretic expressions is “anchored” by both formal and natural language **descriptions**.
- **Example. Button-style Internalism.** Our language gains its meaning via the mastery of a formal theory, and determinacy is secured via the provision of an **internal categoricity** result.

- There might be an **interplay** here.
- As an example Penelope Maddy's position in *Defending the Axioms* [Maddy, 2011] and Luca Incurvati in *Conceptions of Set and the Foundations of Mathematics*.

- Maddy advocates sets as marking the “contours of mathematical depth”.

*...our set-theoretic methods track ... these underlying contours of mathematical depth. Of course the simple answer — they track sets — is also true, so what we've learned here is that what sets **are, most fundamentally**, is markers for these contours, what they are, **most fundamentally**, is maximally effective trackers of certain strains of mathematical **fruitfulness**. [Maddy, 2011, p. 82]*

- Incurvati advocates that we proceed by “inference to the best conception”

*we should regard the iterative conception as correct because it is the **most satisfactory** conception of set that we have — that is, because it is **better** than its rivals with regards to certain **virtues** that a conception of set may have. [Incurvati, 2020, p. 65]*

- **Prima facie** (and I *emphasise* that it's prima facie, both Maddy and Incurvati tell their own stories) it's **puzzling** why their methodologies should be **truth-conducive**.
- But under some metasemantic theories it **makes sense**.
- We start with some **descriptions**, which circumscribes some **truth**.
- There are then **choicelike justifications** leading to **new** descriptions, **new** truths, etc.
- Given an appropriate metasemantics, it's possible for what's (determinately) true to **change**.

- We've seen that justification is a bit more complex than the crass picture, it can be **choicelike** as well as **truthlike** and it can interact with our views on **metasemantics** and **truth**.
- **But note:** Just because a justification isn't (yet) truthlike given your metasemantic picture, doesn't mean that it's not **objective**.
- There might be **objective** facts about what the **best** theory is for some purpose, even if your ontology and metasemantics doesn't yet support its **truth**.
- We just have to be very **aware** of what we're doing when we try to justify.

Thanks for listening.

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