Metasemantics and the Continuum Hypothesis

Neil Barton

Slides available via the “Blog” section of my website

https://neilbarton.net/blog/

NUS National University of Singapore

- In 1874 Georg Cantor proved that there are more reals than natural numbers — the reals are uncountable.
- An 1878 paper contains his now infamous:

The Continuum Hypothesis.

(CH) Every infinite class of real numbers can be either put in one-to-one correspondence with (i) the natural numbers, or (ii) all the reals.

- We continue to spill ink (and toner) over the Continuum Hypothesis, with different projects pulling in different directions.

- But what do we mean when we utter the terms "Continuum Hypothesis", "set", "membership", and "class".
- One tempting idea is that there's some platonic universe of sets, and we can refer to it.
- This leads to:

The Similarity Assumption

The Continuum Hypothesis and associated terms like "set" and "class", determinately have the *same content* between different agents (you, me, Cantor, Gödel). Agents are disagreeing about the truth or falsity of this content.

Target.

1. Argue that the **Similarity Assumption** is false (or at least probably wrong), especially on one popular "moderate" way of construing metasemantics.
   BUT
2. Some versions of CH are probably determinate.

- First, what is metasemantics?
- Metasemantics is the study of how our utterances (thoughts etc.) get their meanings or content.
- For example, perhaps you're a gödelian; you think we have faculty of intuition that we can use in attaching meaning to claims about sets and membership etc.
- Or perhaps you're a inferentialist; you think that our assertions get content in virtue of their inferential role.

- I don't want to commit to any one metasemantic view.
- I'll nestle between the following two extremes:

Strict metasemantics

Content has little to do with an individual agent or minority community's patterns of usage or practice. Rather, content is fixed by large-scale global and/or agent-external features.

- **Examples.** Lewis, gödelian platonists.

Lax metasemantics

(i) As long as an agent A's practice is consistent, it's true of some entities or other, and the content of A's thoughts and assertions can reasonably construed as about those entities, and

(ii) Our practice is not capable of determining content concerning an infinitary subject matter.

- **Examples.** Extreme conventionalists, JDH.

Moderate metasemantics.

1. It is not the case that some practice automatically has content that is determinate (no strong reference magnets etc.)
2. but nor is our practice an “anything goes” affair incapable of determining infinitary content.

- For a moderate, our practice can make infinitary content determinate once we become sufficiently precise.
- **Examples.** Soysal-style descriptivism and anchors, Kreiselian informal rigour, Button and Walsh's internalism, even Warren-style conventionalism.
- **For later.** A moderate idea: The roles (e.g. inferential, justificatory) that terms play in our theorising affect meaning.

- Moderate metasemantics is a broad family.
- Whether content is determinate (and what content is thereby determined) may depend on your view.
- This diversity makes it hard to make uniform claims.
- My suggestion: Look at markers for determinacy.

Markers.

A marker for determinacy is something that, if you've got it, increases my confidence in determinacy and, if you're missing it, my confidence decreases.

We'll use arithmetic as our best candidate for a determinate practice.

Intuitive description.

There should be a reasonably clear intuitive description of the intended underlying structure / domain.

- Some (e.g. Feferman) think this is sufficient for determinacy of arithmetic.
- Others think it sufficient for determinacy of set theory (e.g. Gödel).
- Nothing so strong going on here!

Categoricity.

There should be a categorical description of the structure we intend to talk about in our practice.

- PA_2 is categorical.
- Some moderates (e.g. Button) think this is sufficient.
- But we needn't go so far.
- A (provable absence of a) categoricity theorem tells you that any "clarity" you have can(not) be manifested.

- The next requires a bit of set up.
- Two theories T_1 and T_2 are bi-interpretable when each can interpret the other, and each theory can tell that composing interpretations yields an isomorphism.
- The thought behind bi-interpretability is that if two theories are bi-interpretable then they can each "simulate" the other.
- A deductively-closed theory T is tight when no two non-identical deductively-closed extensions of T (in the same language as T) are bi-interpretable.
- Adding axioms is substantive.

Tightness.

The first-order theory underlying our practice should be tight.

Theorem.

[Visser, 2006] PA is tight.

- Finally we have:

Theoretical Completeness.

Our first-order theory should exhibit a high degree of theoretical completeness (i.e. the theory either proves or refutes most of the sentences it can formalise, particularly the natural ones).

- PA is very theoretically complete.

- In the paper, there's a whole long section on tracing some history.
- TLDR version: Our set-theoretic practice has changed over time.
- Particularly interesting is to think about how the following features have been incorporated.

- **Strong iterative conception.** Roughly: Sets are successively formed via iterating powersets (and collecting via union).
- **Weak iterative conception.** New sets are formed from old via some set-forming operations.
- **Maximality.** There should be "as many sets as possible".
- **Historical observation.** Looking at Gödel's thought, you can see aspects of each coming in to his practice.

- Where are we now? A sketch (more soon):
- **Liberalised Construbtibilism.** (Woodin et al) CH is true, set-formation is weak (though strong conception could be recovered).
- **Forcing-saturated uncountabilism.** (Bagaria et al) CH is false, set-formation is strong.
- **Austere countabilism.** (Meadows, Steel) CH not formalisable, set-formation is weak.
- **Opulent countabilism.** CH is true, set-formation is weak.

- Things to note:
- (1.) Importance of weak set-formation.
- (2.) Commonality of a conceptual ancestor with **Maximality** in the picture.
- (3.) For uncountabilists, well-motivated options for both CH and $\neg\mathsf{CH}$.

- Let me say a little more about the countabilist options.
- **Austerity.** No talk of classes! Just ZFC^- (plus "Every set is countable", abbreviate this by a c prefix, e.g. cZFC^-)
- **Opulence.** You can have as much ~~cake~~ well-motivated class-theory as you like (at least cNBG^-, but have cMK^- too).

- Note that under countabilism, you can still get inner models of ZFC (basically, you can do all your inner model theory).
- Most countabilist systems on the market get ZFC (e.g. [Scambler, 2025], [Barton, MS]).
- You can get a lot more (e.g. $0^\sharp$ [Barton and Friedman, F], or under PD yet more still, Woodins etc.).

- Under austerity, the best we can cobble together are measly local versions of CH in models that have enough "powersets" to support it (e.g. inner models of ZFC).
- Given opulence (and a theory like NBG^- or MK^-) we can also get a version of CH for reals(ies)...

Countabilist-CH

Every class is either bijective with some (countable) set or the universe.

Fact.

Countabilist-CH is equivalent (modulo) NBG^{-} to the Strong Well-Ordering Principle that the universe is well-orderable with order-type *Ord*.

Fact.

The countabilist has an excellent justification for Countabilist-CH; it is equivalent to the Limitation of Size Principle that all proper classes are bijectable.

- Clearly the local versions of CH do not mean the same thing as any global versions of CH (not about all the reals, for a start).
- But I contend that Countabilist-CH and Uncountabilist-CH also mean different things.

- **First.** Are the two versions of CH determinate for each?
- Everybody has: **Intuitive Description**, **Categoricity**, **Tightness** (see [Enayat, 2016]).
- The issue is **Theoretical Completeness**.
- The countabilist has strong motivation for Countabilist-CH, a simple theory of cardinality, if they can motivate PD they get very high theoretical completeness.
- The uncountabilist has problems with **Theoretical Completeness**, in particular motivations for both CH and $\neg$CH

- So: If we take **Theoretical Completeness** to convince us of indeterminacy, then we're done (one content is indeterminate, the other determinate).
- But if both Uncountabilist-CH and Countabilist-CH are determinate, it's more complicated.
- Time to deal with the natural objection...

- **UNCOUNTABILIST:** YOU'RE JUST TALKING ABOUT $H(\omega_1)$, YOU'RE JUST TALKING ABOUT $H(\omega_1)$,...
- **Correct.** If you're an uncountabilist.
- If you're a countabilist, it is the uncountabilist who makes restrictive claims [Barton, 2025].
- To make this stick, we need a meaning-preserving translation between Countabilist-CH and Uncountabilist-CH.
- **Note:** No homophonic translation.
- What about mapping the countabilists first-order domain to $H(\omega_1)$ in the natural way (classes to sets of reals)?

- A principle of moderation (recall Soysal/Warren):

Smoothness.

If ϕ is a sentence in a language $\mathscr{L}_1$, and ϕ^t is a translation of ϕ into a different language $\mathscr{L}_2$, then ϕ^t should have similar anchors, inferential roles, and relevant justifications regarding translations in $\mathscr{L}_2$ as ϕ does in $\mathscr{L}_1$.

- The natural translation from Countabilist-CH to Uncountabilist-CH does not respect **Smoothness**!
- e.g. the justification as a limitation of size principle (clearly doesn't transfer), not a global well-order principle.

- **Wrapping up:**
- Lots more in the paper! Version on PhilArchive to be replaced soon!
- e.g. Could there be some sort of equivalence thesis between the two? (Answer — with the help of Ali Enayat and an anonymous reviewer — "**No!**".)
- Two closing observations:

- (1.) Supposing that both Uncountabilist-CH and Countabilist-CH are determinate, we have a kind of pluralism that allows for determinacy.
- We just have to be clear to disambiguate.

- (2.) The debate on CH and whether there are uncountable sets looks like a verbal dispute (in Chalmers sense).
- But there are other ways to disagree.
- e.g. It may be that the disagreement amongst uncountabilists over whether Uncountabilist-CH holds is semantically conservative whereas the debate over whether there are uncountable sets at all is semantically progressive [Belleri, 2021].
- Maybe what we need in the end is set-theoretic activism.

To the barricades!

Thanks for listening.

References I

Barton, N. (2025).
Is (un)countabilism restrictive?
Journal of Philosophical Logic, 54(5):1061–1077.

Barton, N. (MS).
What makes a 'good' modal theory of sets?
Manuscript under review. Preprint: https://philarchive.org/rec/BARWMA-9.

Barton, N. and Friedman, S. (F).
Countabilism and maximality principles.
To appear in the *Notre Dame Journal of Formal Logic*. Preprint: https://philpapers.org/rec/BARCAM-5.

Belleri, D. (2021).
Downplaying the change of subject objection to conceptual engineering.
Inquiry, 0(0):1–24.

Enayat, A. (2016).
Variations on a visserian theme.
In van Eijck, J., Iemhoff, R., and Joosten, J. J., editors, Exploring the Frontiers of Incompleteness, pages 99–110. College Publications.

References II

Scambler, C. (2025).
On the consistency of height and width potentialism.
In Antos, C., Barton, N., and Venturi, G., editors, The Palgrave Companion to the Philosophy of Set Theory, pages 305–331. Springer Nature Switzerland, Cham.

Visser, A. (2006).
Categories of theories and interpretations, page 284–341.
Lecture Notes in Logic. Cambridge University Press.