Mathematical Contingency

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November 14, 2025

The Continuum Hypothesis

This talk is about whether the mathematical world could have been different.

Here is the continuum.

The red subset is equinumerous with the entire continuum.

The green subset is equinumerous with the natural numbers.

Cantor's *continuum hypothesis (CH)* is the claim that every infinite subset is like that: either equinumerous with the whole continuum or with the naturals. I.e., there are no subsets of intermediate size.

It turns out CH is independent of the usual axioms of mathematics.

This is shown with the technique of *forcing*. Forcing is a way of generating new models of mathematics from old ones. With forcing we generate models with or without CH by adding different kinds of *subset*.

To force CH to become false, we "add new real numbers in between the old ones".

To force CH to become true, we "add new one-to-one correlations between all the intermediate-sized subsets and the natural numbers".

Forcing and Contingency

CH Contingentism about CH (Contingentism): The world could have been a forcing extension of how it actually is, such that the truth about CH would have been different.

Variability of the continuum: there could have been more real numbers than there are. (Equivalently: there could have been more subsets of the natural numbers than there are.)

Variability of bijection: there could have been more bijections between collections than there are.

CH Contingentism and Physical Reality

Does Contingentism make any sense?

The continuum is interesting because its structure is mirrored in things that matter to us.

Example. Physical space. Contingentism seems to say that if spacetime kept its actual structure, gaps would appear in it.

Example. Possibility space. The possible outcomes of an infinite sequence of coin tosses are thought to be structurally the same as the sets of natural numbers. Contingentism seems to say there could have been new possibilites regarding how the coin might land.

CH Contingentism and Physical Possibilities

An informal argument against new reals. Imagine God necessarily tosses coins indexed by natural numbers c_0, c_1, etc. Say a class of natural numbers X is *selected* by the coin tosses if c_i lands heads iff $i \in X$.

- Necessarily, any class of natural numbers could have been selected.
- So if there was a new class of natural numbers, it could have been selected.
- Go to a possible world w where that happens. Then let X be the class of numbers n such that c_n lands heads in w.
- So the selected class at w is X—but that's not a new class after all. Contradiction!

CH Contingentism and Recombination

The argument had two main assumptions. First there is the way God's coin works. What we need is:

Recombination. There is a property of natural numbers X such that necessarily, for every class of natural numbers, C, it is possible for C to be the extension of X.

The other assumption was that you can find a possible world where any possibility obtains:

Atomicity. Necessarily, every possibly true proposition is compatible with a propositional atom (i.e., to be possible is to be true at some possible world).

Questions

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Is the argument from Atomicity + Recombination against Contingentism valid?

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Are Atomicity and Recombination individually consistent with Contingentism?

Previous Work on CH Contingentism: Potentialism

Several views (e.g. Hamkins, Scambler) put forward the idea that there could have been more sets, including more sets of natural numbers/bijections.

However, their frameworks only quantify over rigid collections, individuated extensionally.

This seems restrictive when considering claims about possibility.

For example: they can't easily make sense of the Atomicity + Recombination argument.

Previous Work on CH Contingentism: Bacon

Bacon uses a higher-order logic where you can talk about properties: *Classicism.*

Classicism is classical higher-order logic, where instead of a rule of extensionality we have a rule of intensionality: from proving '**P** iff **Q**' from no non-logical assumptions you can substitute **P** and **Q** in any context.

Bacon proves the consistency of some extensions of Classicism with the contingency of CH.

However, Bacon's systems are missing a way of talking about the extensions of properties. Phrases like 'the actual real numbers' do not make sense in his systems.

So Bacon also cannot assess our informal against Contingentism.

Classicism + $\Box$Rigid Comprehension

We want a system that allows us to quantify over both rigid and non-rigid properties.

Bacon and Dorr's **Classicism** handles reasoning about properties and propositions and modality.

Rigid Comprehension says every property is coextensive with a modally rigid one (i.e., it's extension).

Our system is Classicism + $\Box$Rigid Comprehension. So we can reason modally variable properties and their modally rigid extensions in the same system.

Theorem

In Classicism + $\Box$Rigid Comprehension, Atomicity + Recombination together entail there cannot be new real numbers or new bijections between naturals and sets of reals. So Contingentism is refuted in Classicism + $\Box$Rigid Comprehension + Atomicity + Recombination.

The Consistency of CH Contingentism

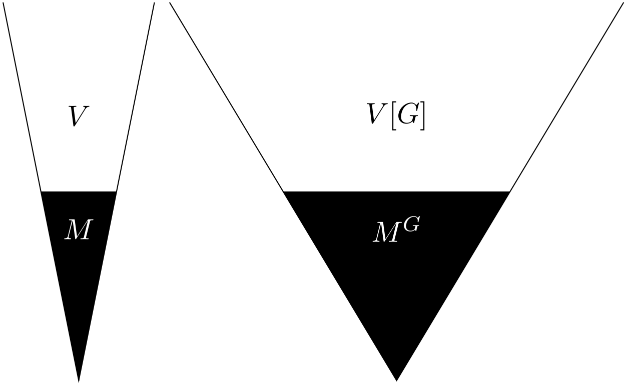
Theorem

Contingentism is consistent with Classicism + $\Box$Rigid Comprehension + Atomicity, and with Classicism + $\Box$Rigid Comprehension + Recombination.

Proof Strategy

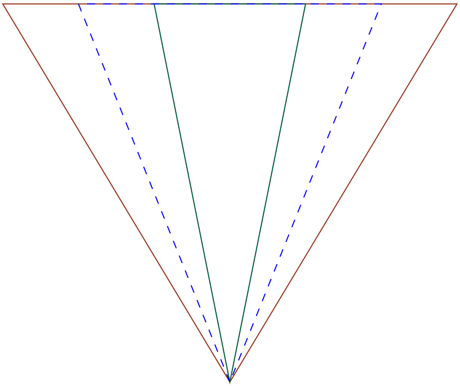
Work in a model of set theory V with a forcing extension $V[G]$.

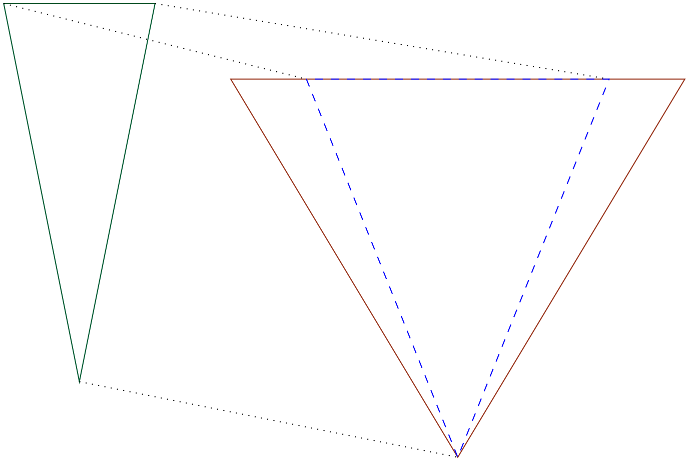
Pick some model construction for Classicism. Run it twice, to get the model M in V and the model M^G in $V[G]$. $M \subseteq M^G$.



Proof Strategy

M embeds into M^G. N^G is the set of things in M^G definable (in the language of HOL) from parameters in M.





In our model, the actual world is like M except we have added a new possibility to the model.

There is one other possibility: things are as in M^G.

Each thing in the actual world becomes a thing in N^G. Things in M^G but not N^G are things that will exist but don't exist yet.

This gives us a model of Classicism + $\Box$Rigid Comprehension. By varying the construction of M we can add enforce either Atomicity or Recombination.

Crucially, this whole construction can be carried out inside V by encoding members of N^G as expressions in the language of HOL with parameters from M.

For this, we need to use a forcing for which V can tell what will be true in M^G (a V-decisive, or equivalently weakly homogeneous, forcing).

Since the model is like M in the actual world, it satisfies CH iff V does. Since it is like M^G in the other possibility, we can make CH contingent by making $V[G]$ differ from V on CH. Done!

Generalisations

This method can be used to make CH flip truth value any finite number of times.

But contingentists generally think CH is *necessarily* contingent. Our method doesn't generalise to that case without more work.

Independently: we used weakly homogeneous (V-decisive) forcings to make sure the model could be constructed in V. But we think the method can be generalised to arbitrary forcings too.

The End!