Perspectivism and Semantic Content: Groundwork for a Mathematical Perspectivism

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- At some level, every mathematician is a pluralist.
- The "correct" theory is the one that will yield a proof of an especially recalcitrant theorem or deliver understanding.
- There are deeper philosophical questions about pluralism. Is a particular theory true? Is there even a unique true theory?
- But generally speaking, if a mathematician can adopt a different perspective for some end, then they'll take it.

- This hasn't stopped a debate going on between advocates of set-theoretic and category-theoretic foundations.
- Each accuses the other side of committing various philosophical and mathematical sins.
- Others (e.g. me) suggest a more pluralistic outlook, on which each should be understood as about a different subject matter.

Main aims.

1. Argue for a new kind of position in the philosophy of mathematics: Perspectivism.
2. Get some feedback on notions used in the paper (in particular regarding notions of semantic sameness).

Introduction

TLDR: Set theory and category theory

Pluralism and Perspectivism: An interlude from the philosophy of science

Enter metasemantics

Pre-conclusion: The doors of perspectivism are open

- We'll be assessing theories qua foundations.
- For me, a foundation comprises:
  (1.) A *formal language.*
  (2.) A (family of) *axiomatic theory (or theories)* in that language.
  (3.) A rough-and-ready description of what that language is *about* — how the language should be *understood* — and, in particular, why the axioms are *justified.*

- There's various desiderata we'd like from a foundation (discussed recently by Penelope Maddy).
- The ones relevant for us:
- **Generous Arena.** Find *representatives* for our usual mathematical structures (e.g. the natural numbers, the real numbers) using our foundational theory.
- **Metamathematical Corral.** Provide a theory in which metamathematical investigations of relative provability and consistency strengths can be conducted.
- **Risk Assessment.** Provide a degree of confidence in theories commensurate with their consistency strength.
- **Essential Guidance.** "...capture the fundamental character of mathematics as it's actually done, that will guide mathematicians toward the truly important concepts and structures, without getting bogged down in irrelevant details." [Maddy, 2017, p. 305]

- **Set theory:** Single primitive $\in$, various axiomatisations trying to get at the universe(s) of sets.
- **Category theory:** Idea of arrow, domain and co-domain, good for systematising algebraic notion of what it takes to be a map and isolating notions of structure/information preservation.

- TLDR version:
- Set theory is great for **Generous Arena**, **Metamathematical Corral**, and **Risk Assessment**, but is bad for **Essential Guidance**.
- Category theory is great for **Essential Guidance**, can do **Metamathematical Corral**, and **G**enerous Arena, gives us no Risk Assessment.
- **Claim:** They are dual because of what they are about.
- So: **Obviously** we should just be pluralists — they are about different things (cf. [Barton and Friedman, 2019]).

- The arguments above can be given some mathematical details (ask if interested).
- But what I really want to do is get the notion of perspectivism on the table, and what we might expect in a mathematical context.

- I'll set up what I mean by the distinctions:

Pluralism.

Pluralism about a given field F is the position that there are multiple legitimate theories concerning F (in our case these will be foundational frameworks).

Strong pluralism.

A view is strongly pluralist if, in addition to pluralism, it holds that there is significant incommensurability between the (natural interpretations of) the two theories.

Perspectivism.

We should be pluralists, but only insofar as our frameworks provide alternative perspectives on the same subject matter.

- Example of strong pluralism: Nancy Cartwright's "dappled world" (e.g. Hooke's Law in beam mechanics).
- Example for perspectivism: Giere's example of dichromates vs. trichromates.
- Example for perspectivism: Newtonian, Hamiltonian, and Lagrangian formulations of classical mechanics.
- Equivalent in some sense (unclear to me exactly which) but empirically equivalent.
- Getting at a bunch of the same structure, but slightly differently embedded.
- Quantum vs. relativistic mechanics?

- **Suggestion.** The strength of the pluralism involved is a matter of degree.
- It's not a case of being pluralist or not, it's rather a case of how different does the target look.

- Contrary to my old self, I think there's some space for a perspectival approach to mathematical foundations.
- There's some mathematical / formal properties here, that I will suppress (gory details at the end for those interested).
- But I think there's a philosophical way in.

- The core idea is that of metasemantics.
- How do our assertions (and thoughts etc.) get their content?
- **Option 1.** (Strict) Everything is determinate, because magic. (Lewis, many interpretations of Gödel.)
- **Option 2.** (Lax) *Anything* (first-order) you assert, if consistent, determines some subject matter.
- **Option 3.** (Moderate) (Soysal, Warren, Me(?)) Content is determined by the thoughts and patterns of use of individual agents.

- For strict and lax approaches, there's no interesting perspectivism.
- Strict: Everything you say has fixed content, no scope for perspectivism.
- Lax: You only talk about the same thing when you assert equivalent theories.

- For moderates there's the possibility of selecting what you talk about.
- I talk about natural numbers as though they are kinds of set.
- You talk about natural numbers as though they are particular operations.
- The content isn't exactly the same, because we have different patterns of use.
- But it can have overlap or be shared.

- It's natural to think that different foundations are just directed at different subject matters.
- But metasemantics reveals that there might be a different option; slightly different content, but with some shared and overlapping subject matter.
- **Options.**
  1. Stop here to discuss.
  2. Start seeing some gory mathematical details supporting the idea that the difference is perspectival rather than strongly pluralist.

- Two things we should want to convince ourselves that the target is the same.

Translation.

In order for the difference between two frameworks to be considered *perspectival*, there should be mutual interpretability between the relevant formal theories. An increase in “similarity” between the different perspectives (e.g. witnessed by an increase in the robustness of the interpretability phenomenon), indicates a decrease in the strength of pluralism being offered.

Interaction effects.

The existence of interaction effects is evidence that the difference between two frameworks is more perspectival and less strongly pluralist.

- We'll consider the following candidate category theories:
  - CCAF (the *Category of Categories as an Autonmous Foundation*) has the following axioms:[1]
  - Existence of finite categories **1**, **2**, and **3**.
  - Arrow extensionality.
  - The existence of products and coproducts for any two categories.
  - Every parallel pair of arrows (functors) has an equalizer and a co-equalizer.
  - For any two categories, the functor category between them exists.
  - There is a natural number category.
  - Categorial choice.

[1] This is adapted from Chapter 9 of [McLarty, 2008].

- ETCS^+ is contains the axioms of a well-pointed topos with a natural number object and satisfying (i) a categorial version of the Axiom of Choice (epics split), and (ii) a categorial version of the axiom of replacement.
- CCAF^+ is CCAF plus the axiom that there is a category satisfying ETCS^+, plus the Fullness principle that: For all sets $I, J : \mathbf{1} \to \mathbf{Set}$, every functor $(1 \downarrow I) \to (1 \downarrow I)$ equals $(1 \downarrow I)$ for some function $f : I \to J$ in $\mathbf{Set}$.

- We'll consider the following set theories:
- ZFC
- (That's all apart from the odd inaccessible for flavour.)

- Two theories are bi-interpretable (a pretty robust kind of theoretical similarity), when you can define translations each way, such that composing translations gets you something isomorphic with what you started with.
- A tweak:

Definition.

Two theories T_1 and T_2 are χ-*interpretable* when there are interpretations $F : \mathsf{T}_1 \to \mathsf{T}_2$ and $G : \mathsf{T}_2 \to \mathsf{T}_1$, such that:

(i) If $M \models \mathsf{T}_1$, then $G \circ F(M) \equiv_{cat} M$

(ii) IF $N \models \mathsf{T}_2$, then $F \circ G(N) \equiv_{cat} N$.

Theorem

[Mitchell, 1972] [Osius, 1974] (see [Meadows, F] for this presentation) There are interpretations $catset : \mathscr{L}_{cat} \to \mathscr{L}_{\in}$ *and* $setcat : \mathscr{L}_{\in} \to \mathscr{L}_{cat}$ *such that:*

(1.) *If* $M \models \mathsf{ZFC}$, *then* $catset \circ setcat(M)$ *is categorially equivalent to* M.

(2.) *If* $N \models \mathsf{ETCS}^{+}$ *then* $setcat \circ catset$ *is categorially equivalent to* N.

Fact

(Meadows, personal communication.) ETCS^+ *and* ZFC *are not bi-interpretable.*[a]

[a] The result is known, but the subject of current work by Meadows, and so I won't include a description of the proof here.

- So it looks like ETCS^+ and ZFC are different but somewhat perspectival.
- But what about CCAF^+?
- TLDR version: There's some real challenges, but it's open.
- For sure the Osius-Mitchell translation is problematic, basically because of functor categories.
- ZFC and CCAF^+ for sure aren't bi-interpretable, but it's at least possible that there's extensions that are χ-interpretable or stronger.

- **Interaction effects.**
- Isbell's result that $\mathbf{Set}^{Op}$ is bounded iff there's no proper class of measurable cardinals.
- Bagaria and Brooke-Taylor's result on colimits and elementary embeddings.
- TLDR: There are interaction effects, but I'm still to digest them and I don't want to go into too much detail here.
- What I do want to do is raise some issues for what I've tentatively suggested today.

- **What do we think of the notion of semantic preservation (and degrees thereof) at play?**
- e.g. as witnessed by formal theories.

- **What about interaction effects?**
- Are they indicative of being in a perspectival state?

- **Is the notion of degree of perspectivism cogent?**

- **How should we handle inconsistent information?**
- e.g. via Brown and Priest's chunk-and-permeate?
- Paraconsistent logic?
- Tools from the perspectivism literature (e.g. in Michaela Massimi's work)?

- **What of actualism and potentialism and the Putnam-Button Equivalence Thesis?**
- Should this rather be understood as a close difference, but still with important perspectival features?

- **Can we formalise the notion of perspective to good effect?**

1. What do we think of the notion of semantic preservation (and degrees thereof) at play?
2. What about interaction effects?
3. Is the notion of degree of perspectivism cogent?
4. How should we handle inconsistent information?
5. What of actualism and potentialism and the Putnam-Button Equivalence Thesis?
6. Can the notion of a perspective be formalised to good effect? (And obvious metaperspectivism worries.)

Thanks! Discussion!

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Somewhat incomplete — sorry!

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