Putnam ‘What is mathematical truth?’ (1974)

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Preamble

I hadn’t read this paper since grad school (probably about ten years).

General feeling: It’s *very* interesting with lots of different threads! (Typical Putnam stuff.)

Confession. I found the paper *hard*.

I’ll try to bring out where I’m puzzled and we can discuss further (obviously raise any of your own questions too)!

Main claims

I could detect the following **Main Claims**:

1. Realism (of some kind) is needed for mathematics.
2. We can obtain truth value realism without committing to object realism.
3. (Quasi-)Empirical methods are a key part of discovery in mathematics.

Herein lies some of the difficulty. The paper is entitled ‘What is mathematical truth?’ (this looks it’ll be concerned with metaphysics)

He then *explcitly* says he’s argued primarily for a *practice-based* or *epistemological claim*. From the conclusion:

> In this paper, I have not argued that mathematics is, in the full sense, an *empirical* science, although I have argued that it relies on empirical as well as quasi-empirical inference. (p. 77)

And woven through is an argument about *realism*...

...backed up by an apparent *mathematical point* about equivalence...

A bit of background

This paper appears at a somewhat tumultuous time in the philosophy / foundations of mathematics (especially in the US).

Historically, mathematics went through a revolution of rigorisation in roughly the period 1850 - 1950.

> It was simply assumed that mathematical knowledge would have to be a matter of proof, that is, deduction from the axioms... (Giaquinto, *The Search for Certainty*, p. 5)

Then:

- The 'New Math' had swept the United States starting in the 1950s (see Tom Lehrer's excellent parody).
- Independence of the Continuum Hypothesis published 1963.
- Benacerraf's 'What numbers could not be' published 1965.
- Benacerraf's 'Mathematical Truth' published in 1973.

That's all to say: This paper is occurring at a time of great upheaval in the United States, and I think you can see that in the way it is written.

1 Realism is needed

Key problem when working in this area: It's sometimes unclear what *realism* means.

Carrie Jenkins, for example, isolates *ten* versions of realism in her book *Grounding Concepts* (2008)

Two popular ones:

1. **Object realism.** There is a distinct realm of mathematical objects that is abstract, atemporal, non-causal, etc. etc. ("π in the sky", to borrow a phrase from Jack Woods).
2. **Truth value realism.** Statements of mathematics are *true* or *false* for reasons *external* to humans.

Putnam takes the following version due to Michael Dummett:

> (pp. 69–70) A realist (with respect to a given theory or discourse) holds that:
>
> 1. the sentences of that theory or discourse are true or false; and
> 2. that what makes them true or false is something *external* - that is to say, it is not (in general) our sense data, actual or potential, or the structure of our minds, or our language, etc.

Putnam then professes his 'No Miracles' argument.

> The positive argument for realism is that it is the only philosophy that doesn't make the success of science a miracle. (p. 73)

And argues that this applied in mathematics too (p. 73).

Two supports for this:

1. Mathematical experience. ("The construction of a highly articulated body of mathematical knowledge with a long tradition of successful problem solving is a truly remarkable *social* achievement.")

2. Physical experience (including our scientific theories of the external world).

Problem for 1. What about theology?

Response. Theology was inconsistent.

What if we "fixed it up"? Doubtless true in some finite model?

Puzzlement. I have no idea what the justification for this claim is meant to be.

Still, mathematics certainly seems to deal with infinite models, and at least if you want consistency claims to be part of mathematics, you'll bump up against Gödel's Theorems.

A kind of abductive argument for the consistency of mathematics:

> In mathematics we have (we think) a consistent structure - consistent notwithstanding the fact that no science other than mathematics deals with such long and rigorous deductive chains as mathematics does (so that the risk of discovering an inconsistency, if one is present is immeasurably higher in mathematics than in any other science) (p. 74)

This is an instance of the *quasi-empirical stance* (more on this later), but we might wonder if this argument is any good?

OK: But let's run with it.

What about just "consistent under some interpretation"?

Physical experience shows the realistic one is needed.

Claim. Physics is shot through with mathematics. (p. 74)

Question. Does physical experience really *need* the realistic interpretation?

(Usual Putnamian issues concerning what fixes this etc. no "lost noumenal waifs")

Question. What if the physical universe is finite?

Question. What if we only needed a *countable* model? Would that be "realistic enough" for Putnam?

2 Modal truth-value realism

OK: Let's suppose we assume that we've gotta be truth value realists.

There's a natural backing of Truth-Value Realism, namely Object Realism.

But Putnam doesn't want to take this approach:

> ...mathematics has no objects of its own at all. You can prove theorems about anything you want — rainy days, or marks on paper, or graphs, or lines, or spheres — but the mathematician, on this view, makes no existence assertions at all. What he asserts is that certain things are possible and certain things are impossible - in a strong and uniquely mathematical sense of 'possible' and 'impossible'. In short, mathematics is essentially modal rather than existential, on this view, which I have elsewhere termed 'mathematics as modal logic'. (p. 70)

OK: How does this go?

Rough idea: (Here we dip into Putnam's 1967 'Mathematics without foundations') Rather than committing to existential claims, we:

1. Commit to the existence of a 'mathematical modality'.

2. Prefix existential claims with a $\Diamond$ and universal claims with a $\Box$, relativising to the possible truth of the axioms if necessary.

A couple of questions here:

Question. As it turns out, it's not totally obvious what's the best way cash out this in every case.

Putnam's move works, but might we want this done more 'naturally'? (e.g. for set theory, Parsons, Linnebo, Studd, Button)

Question. Just what on earth is this 'mathematical modality'?

It's not meant to be metaphysical...right?

Putnam identifies Hume's Problem (we only observe what is actual, so how to get traction on what's necessary/possible)?

Question. What is the *gain* here meant to be?

We avoid worries about the causal theory of reference (e.g. Benacerraf), or at least they can be "clarified" but this "goes beyond the burden of the current paper".

Question. Modal notions WTF/H???????????

I just don't see that modal notions are on *any* better ground.

Question. Is there a contrast with the *physical* sciences here?

In other places, Putnam seems to advance something like the following (this formulation is due to Tim Button's 'Level Theory Part 2' and is in a slightly different context):

The Potentialist/Actualist Equivalence Thesis. Actualism and potentialism do not disagree; they are different but equivalent ways to express the same facts.

Putnam does seem to come close at points:

> Given that one can either take modal notions as primitive and regard talk of mathematical existence as derived, or the other way around... (p. 72)

> In mathematics, the different equivalent formulations of a given mathematical proposition do not call to mind apparently incompatible pictures as do the different equivalent formulations of the quantum theory, but they do sometimes call to mind radically different pictures, and I think that the way in which a given philosopher of mathematics proceeds is often determined by which of these pictures he has in mind, and this in turn is often determined by which of the equivalent formulations of the mathematical propositions with which he deals he takes as primary. ('Mathematics without foundations', p. 47)

Question. Does Putnam support the Equivalence Thesis?

Question. What do *we* think of the Equivalence Thesis?

Question. Don't we need *more* than the truth of the *first-order* axioms to get the 'real' realism?

3 Quasi-empirical methods

Putnam gives a bunch of examples that show that mathematical discovery (or perhaps knowledge) is underwritten not by proof, but by quasi-empirical methods.

At the time this was quite a bold claim!

And Putnam is *right* that mathematical discovery doesn't just proceed by churning out first-order derivations from axioms.

e.g.1. $P \neq NP$

e.g.2. General way in which proofs are discovered (first instances, conjecture, sketch, proof,..., maybe even formal verification).

This supports the idea that mathematics is roughly continuous with science, perhaps shoring up the 'no miracles' argument.

Question. Putnam gives a host of examples. I won't go through them all, would anyone like to discuss some?

Question. Should we accept them?

e.g. verification of instances of the Twin Prime Conjecture or Goldbach's Conjecture.

Question. Just when are these kinds of empirical verification licensed?

e.g. Rubbish even for $\mathbb{N}$ when the counterexample could be really big.

Also though, we should distinguish between *discovery/justification/knowledge of a mathematical proposition* and *mathematical justification* (which is something like proof).

The role for each may be different.

To my mind, Putnam seems a little stuck in the following view:

Mathematics as a propositional knowledge generator. The primary function of mathematical research is to generate knowledge *that* propositions are true.

But I think this isn't right.

Part of the reason we do mathematics is to get *understanding*.

And I don't see that the quasi-empirical methods deliver *that*.