Potentialist Sets, Intensions, and Non-Classicality

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I thought I'd start with some hot-off-the-press philosophy:

> *something is infinite if, taking it quantity by quantity, we can always take something outside. (Aristotle, .Phys. 207A7–15)*

This might be old, but similar intuitions have gained some currency in the philosophy of set theory recently.

> *Set-theoretic potentialism is the view in the philosophy of mathematics that the universe of set theory is never fully completed, but rather unfolds gradually as parts of it increasingly come into existence or become accessible to us. [Hamkins and Linnebo, 2018, p. 1]*

- Alongside potentialism we have the idea of intensions.
- An intension is a collection that can change members (perhaps it's individuated by some condition or other).
- On the 'standard' view where we have an 'unchanging' universe, there isn't much room for intensions (since every intension just identifies a single extension).
- But under potentialism there's the possibility of having interesting intensions.
- How to formalise these ideas?

Main aims.

1. Present some of the tools that potentialists have used (and why they won't work in the intensional context).
2. Suggest a different option, and give some preliminary results about propositional logics.
3. Explain some challenges and open questions moving forward.

- Let's first discuss the different kinds of potentialism on offer.
- **Height potentialism** says that given some 'proper class' or other X, one can turn X into a set (you'll then have new "proper classes").
- **Rank potentialism** is the version of height potentialism where we can always form the powerset of a given domain.
- **Width potentialism** says that given some "powerset" $\mathcal{P}(x)$ or other, one can add new subsets not in $\mathcal{P}(x)$ (you'll then have a "new" powerset of x).
- **Forcing potentialism** is the version of width potentialism where we can always add a generic for any given partial order and family of dense sets.

- It's natural to formalise these ideas modally.
- I won't go into full details, but here's the shape of the formalisation.
- We add a new modal operator $\Diamond$ to the language.
- We define an appropriate modal logic (usually classical $\mathsf{S4.2}$, maybe more) and give axioms for our potentialism in that logic.

- OK, but let's be clear: It would be totally unreasonable to expect mathematicians to adopt these modal theories.
- Thankfully there's a translation from the familiar language of set theory $\mathscr{L}_\in$ into our modal language.
- Again, suppressing details, one basically just replaces $\forall$ with $\Box\forall$ and $\exists$ with $\Diamond\exists$ (this is called the potentialist translation).

We can now prove some nice theorems:

Theorem

[Linnebo, 2013] There is a modal theory of sets for height potentialism that interprets every axiom of ZFC under the potentialist translation.

Theorem

[Scambler, MS] There is a modal theory of height potentialism and width potentialism that interprets every axiom of ZFC^- + "Every set is countable" + $\underset{\sim}{\Pi}^1_1$-Perfect Set Property.

"Theorem"

[Barton, MS] There is a modal theory of width potentialism that interprets ZFC^- + "Every set is countable" + $\underset{\sim}{\Pi}^1_1$-Perfect Set Property.

- This is great for claims about sets.
- We can read the "normal" mathematical quantifiers as implicit modal quantifiers.
- But there's a couple of problems.
- **Problem 1.** We need classical logic, and this is a little controversial (stay tuned).
- **Problem 2.** For intensions, the proofs depend on the following stability axioms:
- (**Positive stability.**) $x \in y \rightarrow \Box x \in y$
- (**Negative stability.**) $x \notin y \rightarrow \Box x \notin y$
- Neither of these looks very plausible for intensions.
- Consider:
  - The intension given by the condition "x is constructible" under height potentialism.
  - The intension given by the condition "x is countable" under width potentialism.

- Two ways of finding stuff out about a subject matter:
  1. Write down some good **axioms**.
  2. Study a bunch of **structures**.
- Option 1 looks dead using current technology (at least for intensions).
- Option 2 shows some promise.

Definition.

[Hamkins and Linnebo, 2018] A potentialist system (for sets) is a pair $\mathbb{S} = (S, \leq_{\mathbb{S}})$, where S is a collection of structures of a certain kind and $\leq_{\mathbb{S}}$ is a refinement of the substructure relation.

- We'll modify this approach to fit intensions.
- First let's see some examples.
- Fix some countable transitive model $M \models \mathsf{ZFC}$.

Definition.

The Rank Potentialist System is the following potentialist system:

(i) $S = \{V_\beta^M | \beta \in On^M\}$

(ii) $V_\alpha^M \leq_{\mathbb{S}} V_\beta^M$ iff $\alpha \leq \beta$.

Definition.

The Woodin Generic Multiverse Potentialist System is the following potentialist system:

(i) $S = \{M[G] | \text{“}G \text{ is } M\text{-generic”}\}$

(ii) $M[G] \leq_{\mathbb{S}} M[H]$ iff $M[H]$ is a forcing extension of $M[G]$.

Definition

Let $M[G]$ be a $Col(\omega, < Ord)$ class forcing extension of M. Then *the Steel Generic Multiverse Potentialist System* is the following potentialist system:

1. $S = \{x | \text{“}x = M[H]$ and there is an $\alpha \in M$ such that $H \in M[G \restriction \alpha]\text{”}\}$
2. $M[H_0] \leq_{\mathbb{S}} M[H_1]$ iff $M[H_1]$ is a forcing extension of $M[H_0]$.

Definition.

The Countable Transitive Potentialist System is the following potentialist system:

(i) $S = \{N |$ "N is a countable transitive model of ZFC containing M"$\}$

(ii) $N \leq_{\mathbb{S}} N'$ iff $N \subseteq N'$.

Definition.

The Actualist (Potentialist) System is the following potentialist system:

(i) $S = \{M\}$

(ii) $M \leq_{\mathbb{S}} M$ iff $M = M$

- We now want to add in intensions.
- Start by considering the full-domain of $\mathbb{S}$, $fulldom(\mathbb{S}) = \bigcup_{W \in \mathbb{S}} W$.
- We'll want to allow new intensions to emerge, so we'll handle this by using partial functions from $\mathbb{S}$ to $fulldom(\mathbb{S})$.
- **Choice point:** I don't want intensions to pop out of existence, so we'll add the following requirement.

Definition.

A partial function $f : \mathbb{S} \to \mathcal{P}(fulldom(S))$ is *upwards defined on* $\mathbb{S}$ if whenever $f(W)$ is defined, $f(W')$ is also defined for all $W' \geq_{\mathbb{S}} W$.

- **Choice point:** We also don't want functions to reach beyond the domains over which they are define. So we'll add the following:

Definition.

An upwards defined partial function $f : S \to \mathcal{P}(\bigcup_{W \in S} W)$ is $\mathbb{S}$-*bounded* iff $f(W) \subseteq W$ for every $W \in dom(f)$.

Definition.

An *intensional potentialist system* is a triple $\mathbb{S} = (S, \leq_{\mathbb{S}}, \mathcal{X})$, where $(S, \leq_{\mathbb{S}})$ is a potentialist system and $\mathcal{X}$ is a collection of upwards-closed $\mathbb{S}$-bounded partial functions for $(S, \leq_{\mathbb{S}})$.

An important example:

Definition

The *definabilist intensional system over* $\mathbb{S}$ is the system that has as intensions all definable intensions at every given world.

(More formally, given any finite list of parameters $\vec{x}$ and formula $\phi(\vec{x}, y)$, where there is a world $W \in \mathbb{S}$ containing each parameter $\bar{x}$ in $\vec{x}$, there is a $\mathbb{S}$-bounded partial function $f : \mathbb{S} \to \mathcal{P}(fulldom(\mathbb{S}))$ such that $f : W \mapsto \{y | W \models \phi(\vec{x}, y)\}$.)

- Some important examples.
- **Def.** We say that f is *constant* if $f(W) = f(W')$ for any worlds W and W' on which f is defined.
- **Def.** We say that f is *set-like* if there is an $x \in fulldom(\mathbb{S})$ such that wherever $f(W)$ is defined, $f(W) = x$.
- **Def.** *Positively stable* (or '*monotone*' or '*monotonic*') iff whenever we have $x \in f(W)$, then for every $W' \geq_{\mathbb{S}} W$, we have $x \in f(W')$.
- **Def.** *Negatively stable* iff whenever we have $x \in f(W)$, then for every $W' \leq_{\mathbb{S}} W$, we have $x \in f(W')$.
- **Def.** *Stable* iff f is both positively stable and negatively stable.
- **Def.** *Capricious* iff for some set $x \in fulldom(\mathbb{S})$ and every world W with $x \in W$, there are $W_1, W_2 \geq_{\mathbb{S}} W$ such that $x \notin f(W_1)$ and $x \in f(W_2)$.

- **Def.** We say that f is the *stabilisation* of g iff whenever there is an $x \in fulldom(\mathbb{S})$ and W such that $x \in g(W)$ then $x \in f(W')$ for every world at which x exists.
- **Def.** We define f', the *component-wise complement of f* as the partial function $f'(W) = W/f(W)$ (and is undefined wherever f is).
- More classes besides (including weakenings) of these notions.

- What we now want is some way of assigning truth values to set-theoretic claims.
- To begin with, let's just treat the propositional case.
- Here we treat every atomic and quantified sentence as a propositional variable and can then talk about how we assign truth-values to these claims using compositional clauses.

Core intuition.

- Each $\mathbb{S}$ gives us a partial order of the worlds under $\leq_{\mathbb{S}}$.
- We can use this partial order to consider particular algebras.
- We then use algebraic semantics to get truth-values for set-theoretic sentences.

Important here is the following distinction:

Liberal potentialism

Liberal potentialists regard the modal truths as unproblematic. In particular, there are modal truths about generative processes in their entirety, including those that cannot be completed...

Strict potentialism

strict potentialism...goes beyond the liberal view by requiring, not only that every object be generated at some stage of a process, but also that every truth be "made true" at some stage. [Linnebo and Shapiro, 2019]

- How should we assign a semantics?
- **Slogan:** Ask not if a sentence is true, but when.
- For liberal potentialism consider the following:

Definition

- Given an intensional potentialist system $\mathbb{S}$, we let the Boolean algebra for $\mathbb{S}$ (or $\mathbb{B}(\mathbb{S})$) be simply the powerset of the collection of all worlds of $\mathbb{S}$.
- Algebraic operations handled exactly as in the powerset.

- We now assign elements of $\mathbb{B}(\mathbb{S})$ via:
- Propositional variables (e.g. atomics and quantified statements): $[\![\phi]\!] = \{W | W \models \phi\}$.
- Compositional clauses handled by the algebraic operations.

Fact.

Because $\mathbb{B}(\mathbb{S})$ is always a Boolean-algebra, the propositional logic for any potentialist system is classical logic.

- What about the strict potentialist?
- The strict potentialist thinks that claims about sets and classes are made true as we move through $\mathbb{S}$.
- This has more in keeping with a intuitionistic/constructive approach.

- We now modify our algebra we want to consider to $Up(\mathbb{S})$: The collection of all upsets of $\mathbb{S}$.
- **Fact:** This is a Heyting algebra (with pseudocomplements and relative pseudocomplements defined as normal).
- Again, we assign semantics via the algebraic operations.
- But here we assign the value of a propositional variable $[\![\phi]\!]$ according to whether ϕ is satisfied at W and ϕ is satisfied at every $W' \geq_{\mathbb{S}} W$.
- **Slogan:** ϕ is true now and forevermore!

- The calibration of the propositional logic then depends on the structure of $\mathbb{S}$.
- In particular, when it is linear we get Gödel-Dummett logic: all classical tautologies of the form $(\psi_0 \rightarrow \psi_1) \vee (\psi_1 \rightarrow \psi_0)$ added.
- This means that the propositional logic we get is stronger for just the rank potentialist system as opposed to any width potentialism.
- But in all cases we see failures of LEM.
- e.g. "x is constructible or x is not constructible" under rank potentialism.
- e.g. Allowing me a parameter $\bar{x}$ for a set x, "$\bar{x}$ is countable or $\bar{x}$ is uncountable" under any kind of forcing potentialism.

- Some conclusions:
- The propositional logic you get depends on whether you're a strict or liberal potentialist (suitably formalised).
- **Unsurprising.** Liberal potentialists get classical logic, strict potentialists get something **more constructive**.
- **More surprising.** The kind of potentialism can affect the amount of classicality you get.
- In particular, rank potentialism is substantially more classical than any kind of forcing potentialism.

- **Key open question.** How to generalise this to first-order theories (possibly two-sorted to allow for intensions)?
- **Objective.** Provide a semantics for first-order (not just propositional) claims, and examine what goes on.
- This looks hard as many classes are not monotonic.
- **Ideas:**
- Algebra-valued models?
- Iemhoff-Paßmann models?
- Lubarsky models?
- Presheaves?

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