Potentialist Sets, Intensions, and Non-Classicality

Neil Barton

Slides available via the "Blog" section of my website
`https://neilbarton.net/blog/`

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> *something is infinite if, taking it quantity by quantity, we can always take something outside. (Aristotle, .Phys. 207A7–15)*

This might be old, but similar intuitions have gained some currency in the philosophy of set theory recently.

> *Set-theoretic potentialism is the view in the philosophy of mathematics that the universe of set theory is never fully completed, but rather unfolds gradually as parts of it increasingly come into existence or become accessible to us. [Hamkins and Linnebo, 2018, p. 1]*

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- On the 'standard' view where we have an 'unchanging' universe, there isn't much room for intensions (since every intension just identifies a single extension).
- But under potentialism there's the possibility of having interesting intensions.
- How to formalise these ideas?

Main aims.

1. Present some of the tools that potentialists have used (and why they won't work here).
2. Suggest a different option, and give some preliminary results about propositional logics.
3. Explain some challenges and open questions moving forward.

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- **Width potentialism** says that given some "powerset" $\mathcal{P}(x)$ or other, one can add new subsets not in $\mathcal{P}(x)$ (you'll then have a "new" powerset of x).
- **Forcing potentialism** is the version of width potentialism where we can always add a generic for any given partial order and family of dense sets.

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- We add a new modal operator $\Diamond$ to the language.
- We define an appropriate modal logic (usually classical $\mathsf{S4.2}$, maybe more) and give axioms for our potentialism in that logic.

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- Thankfully there's a translation from the familiar language of set theory $\mathscr{L}_\in$ into our modal language.
- Again, suppressing details, one basically just replaces $\forall$ with $\Box\forall$ and $\exists$ with $\Diamond\exists$ (this is called the potentialist translation).

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[Barton, MS] There is a modal theory of width potentialism that interprets ZFC^- + "Every set is countable" + $\undertilde{\Pi}^1_1$-Perfect Set Property.

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- Consider:
  - The intension given by the condition "x is constructible" under height potentialism.
  - The intension given by the condition "x is countable" under width potentialism.

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- Option 1 looks dead using current technology (at least for intensions).
- Option 2 shows some promise.

Definition.

[Hamkins and Linnebo, 2018] A potentialist system (for sets) is a pair $\mathbb{S} = (S, \leq_{\mathbb{S}})$, where S is a collection of structures of a certain kind and $\leq_{\mathbb{S}}$ is a refinement of the substructure relation.

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- Fix some countable transitive model $M \models \mathsf{ZFC}$.

Definition.

The Rank Potentialist System is the following potentialist system:

(i) $S = \{V_\beta^M | \beta \in On^M\}$

(ii) $V_\alpha^M \leq_{\mathbb{S}} V_\beta^M$ iff $\alpha \leq \beta$.

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The Woodin Generic Multiverse Potentialist System is the following potentialist system:

(i) $S = \{M[G] | \text{"}G \text{ is } M\text{-generic"}\}$

(ii) $M[G] \leq_{\mathbb{S}} M[H]$ iff $M[H]$ is a forcing extension of $M[G]$.

Definition

Let $M[G]$ be a $Col(\omega, < Ord)$ class forcing extension of M. Then *the Steel Generic Multiverse Potentialist System* is the following potentialist system:

1. $S = \{x | \text{“}x = M[H] \text{ and there is an } \alpha \in M \text{ such that } H \in M[G \restriction \alpha]\text{”}\}$
2. $M[H_0] \leq_{\mathbb{S}} M[H_1]$ iff $M[H_1]$ is a forcing extension of $M[H_0]$.

Definition.

The Countable Transitive Potentialist System is the following potentialist system:

(i) $S = \{N |$ "N is a countable transitive model of ZFC containing M"$\}$

(ii) $N \leq_{\mathbb{S}} N'$ iff $N \subseteq N'$.

Definition.

The Actualist (Potentialist) System is the following potentialist system:

(i) $S = \{M\}$

(ii) $M \leq_{\mathbb{S}} M$ iff $M = M$

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Definition.

A partial function $f : \mathbb{S} \to \mathcal{P}(fulldom(S))$ is *upwards defined on* $\mathbb{S}$ if whenever $f(W)$ is defined, $f(W')$ is also defined for all $W' \geq_{\mathbb{S}} W$.

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An upwards defined partial function $f : S \to \mathcal{P}(\bigcup_{W \in S} W)$ is $\mathbb{S}$-*bounded* iff $f(W) \subseteq W$ for every $W \in dom(f)$.

Definition.

An *intensional potentialist system* is a triple $\mathbb{S} = (S, \leq_{\mathbb{S}}, \mathcal{X})$, where $(S, \leq_{\mathbb{S}})$ is a potentialist system and $\mathcal{X}$ is a collection of upwards-closed $\mathbb{S}$-bounded partial functions for $(S, \leq_{\mathbb{S}})$.

An important example:

Definition

The *definabilist intensional system over* $\mathbb{S}$ is the system that has as intensions all definable intensions at every given world.

(More formally, given any finite list of parameters $\vec{x}$ and formula $\phi(\vec{x}, y)$, where there is a world $W \in \mathbb{S}$ containing each parameter $\bar{x}$ in $\vec{x}$, there is a $\mathbb{S}$-bounded partial function $f : \mathbb{S} \rightarrow \mathcal{P}(fulldom(\mathbb{S}))$ such that $f : W \mapsto \{y | W \models \phi(\vec{x}, y)\}$.)

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- **Def.** *Stable* iff f is both positively stable and negatively stable.
- **Def.** *Capricious* iff for some set $x \in fulldom(\mathbb{S})$ and every world W with $x \in W$, there are $W_1, W_2 \geq_{\mathbb{S}} W$ such that $x \notin f(W_1)$ and $x \in f(W_2)$.

- **Def.** We say that f is the *stabilisation* of g iff whenever there is an $x \in fulldom(\mathbb{S})$ and W such that $x \in g(W)$ then $x \in f(W')$ for every world at which x exists.

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- **Def.** We define f', the *component-wise complement of f* as the partial function $f'(W) = W/f(W)$ (and is undefined wherever f is).
- More classes besides (including weakenings) of these notions.

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Core intuition.

- Each $\mathbb{S}$ gives us a partial order of the worlds under $\leq_{\mathbb{S}}$.
- We can use this partial order to consider particular algebras.
- We then use algebraic semantics to get truth-values for set-theoretic sentences.

Important here is the following distinction:

Liberal potentialism

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Strict potentialism

strict potentialism...goes beyond the liberal view by requiring, not only that every object be generated at some stage of a process, but also that every truth be "made true" at some stage. [Linnebo and Shapiro, 2019]

- How should we assign a semantics?
- **Slogan:** Ask not if a sentence is true, but when.
- For liberal potentialism consider the following:

Definition

- Given an intensional potentialist system $\mathbb{S}$, we let the Boolean algebra for $\mathbb{S}$ (or $\mathbb{B}(\mathbb{S})$) be simply the powerset of the collection of all worlds of $\mathbb{S}$.
- Algebraic operations handled exactly as in the powerset.

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- Compositional clauses handled by the algebraic operations.

Fact.

Because $\mathbb{B}(\mathbb{S})$ is always a Boolean-algebra, the propositional logic for any potentialist system is classical logic.

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- The strict potentialist thinks that claims about sets and classes are made true as we move through $\mathbb{S}$.
- This has more in keeping with a intuitionistic/constructive approach.

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- **Slogan:** ϕ is true now and forevermore!

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- This means that the propositional logic we get is stronger for just the rank potentialist system as opposed to any width potentialism.
- But in all cases we see failures of LEM.
- e.g. "x is constructible or x is not constructible" under rank potentialism.

- The calibration of the propositional logic then depends on the structure of $\mathbb{S}$.
- In particular, when it is linear we get Gödel-Dummett logic: all classical tautologies of the form $(\psi_0 \to \psi_1) \vee (\psi_1 \to \psi_0)$ added.
- This means that the propositional logic we get is stronger for just the rank potentialist system as opposed to any width potentialism.
- But in all cases we see failures of LEM.
- e.g. "x is constructible or x is not constructible" under rank potentialism.
- e.g. Allowing me a parameter $\bar{x}$ for a set x, "$\bar{x}$ is countable or $\bar{x}$ is uncountable" under any kind of forcing potentialism.

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- The propositional logic you get depends on whether you're a strict or liberal potentialist (suitably formalised).
- **Unsurprising.** Liberal potentialists get classical logic, strict potentialists get something **more constructive**.
- **More surprising.** The kind of potentialism can affect the amount of classicality you get.
- In particular, rank potentialism is substantially more classical than any kind of forcing potentialism.

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