Countabilism: A Survey

Neil Barton
Slides available via the "Blog" section of my website
https://neilbarton.net/blog/

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- I'll be back.

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Main Aims.

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Main Aims.

1. Present some motivations for countabilism from different sources.
2. Identify where interesting points of contact with the group might be for future work.

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- There's some trivial examples of this (e.g. various species of ultrafinitism).
- I want a version that at least allows the existence (again, in some form) of an infinite set.

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- This seems innocent enough to me, but maybe I'm smuggling something in.
- This immediately puts you in the space of second-order arithmetic, and so...

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  - Impredicative math (above Π^1_1-Comprehension).
- Importantly also: Linnebo-Shapiro modal predicativism (more on this later)...

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- A small c prefix, will be used to denote the addition of Count, so cZFC− is ZFC− + Count.

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- This is provably equivalent to the DC-scheme; the scheme of assertions claiming that for each formula $\phi(x,y,z)$ and parameter a, if for every x there is a y such that $\phi(x,y,a)$ holds, then there is an ω-sequence $\langle x_n | n \in \omega \rangle$ such that for all n, $\phi(x_n, x_{n+1}, a)$ holds. (i.e. If a definable relation has no terminal nodes, we can make ω-many dependent choices on its basis.)

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- The $\underset{\sim}{\Pi}^1_1$**-Perfect Set Property** ($\underset{\sim}{\Pi}^1_1$-PSP) is the claim that every Π^1_1-definable class of reals (i.e. countable sets) has the perfect set property.

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- **Projective Determinacy** or PD will be rendered as the schema asserting that every definable class of reals has a winning strategy.
- **Side note (but important)!** If you go to a class theory, the **Limitation of Size** principle that all classes are the same size is equivalent to CH!

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(Work of Zarach, Gitman, Hamkins, Johnston, S. Friedman)
$\mathsf{cZFC}- \not\Rightarrow \mathsf{cZFC}^- \not\Rightarrow \mathsf{cZFC}^-_{Ref}$

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Fact.

$\underset{\sim}{\Pi}^1_1$-PSP $\not\Rightarrow$ PD over any of the above theories (converse is immediate).

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Fact.

Over cZFC^-, PD is equiconsistent with ZFC + "Lots of Woodin cardinals" (in fact PD is equivalent to many 'nice' inner models of ZFC with Woodin cardinals).

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- For now I'll just mention some non-modal axioms.

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- **Forcing Saturation Axiom.** (FSA) There is a generic for any partial order and set-sized family of dense sets.
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- **Ordinal Inner Model Hypothesis** (OIMH) is the EIMH restricted to ordinal parameters.

Some results:

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Fact (also neat exercise).

The FSA and Count are equivalent modulo ZFC−.

Fact.

FSA $\not\Rightarrow$ ASGA, but they are equiconsistent (with ZFC−).

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The EIMH is inconsistent with cZFC^{-}_{Ref}.

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Fact.

The OIMH implies that $0^\sharp$ exists.

Fact.

You need some impredicative class theory to formulate both the EIMH and OIMH.

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- (This is no different from reflection principles in a sense—see the Koellner paper on this—you go pretty quickly from second-order reflection to inconsistency.)
- Is there a different way we might motivate strong axioms?

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- An idea myself and Chris Scambler have been toying with: Take the real numbers as your main foundational object.
- These are the kinds of things that can be used to measure (possible?) spatial magnitudes.
- This reverses the usual 'foundational' arrow.

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- This turns an objection to Freiling's darts (made in the ZFC-context) on its head.
- Our 'foundation' would then be $\mathsf{SOA} + \mathsf{PD}$.
- And the set theory that could thereby be interpreted would be cZFC^{-}_{Ref} (I'm happy to throw Dependent Choice too, but this is a bolt-on) $+ \mathsf{PD}$.

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- (Linnebo-Shapiro) **Modal predicativism.** Clearly countabilist (at least allowing classes) but what one gets depends on a bit on whether strict or liberal. **Pause for discussion.**

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  - The necessitation of the axioms of ZFC.
- **Imprecise point.** Øystein, Sam Roberts, and myself have been playing with some axioms that just assume that there's a 'genie' that can only do 'countable work'. How does this compare? **Pause for discussion.**

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Scambler's (new!) system interprets $\mathsf{cZFC}-$ + whatever Replacement/Collection/Reflection (henceforth RCR-sauce) you're willing to stick in, and is mutually interpretable with $\mathsf{cZFC}-$ + RCR-sauce.

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Adding in a separate modality for height vs. width, and asserting the Linnebo axioms for the height modality gets you mutual interpretability with $\mathsf{cZFC}-$ + RCR-sauce + $\undertilde{\Pi}^1_1$-PSP (see Scambler's contribution to the *Palgrave Companion to the Philosophy of Set Theory*)

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Observation.

There may be sets floating around that aren't obtained by forcing or collapsing, but still get in to a Kripke frame for the axioms...

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- **Reflection** is also a kind of extension of .2, depending on the setting.
- And **Reflection** is non-modally equivalent to the DC-Scheme, so...
- All of these assertions are floating around extensions of .2 or choice-like ideas...(not sure what to make of this, **pause for discussion**).

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- I'm not hopeful of making progress (beyond the isolation of the different interesting 'restrictiveness' notions).

- **Challenge 1.** How to interpret 'standard' math at the level of third-order analysis? (Discussed in the booklet 'Iterative Conceptions of Set' a bunch.)
- **Challenge 2.** Is this a restrictive perspective?
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- I'm not hopeful of making progress (beyond the isolation of the different interesting 'restrictiveness' notions).
- I do think that these discussions show that the notion of cardinal size is very auxiliary assumption dependent.

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- **Option II.** Take the modal theories to be giving you a version of the iterative conception (advanced in the booklet).

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- It seems to me that a possible bolt-on is the assumption that every set can be countablised at some point of the process.
- Perhaps this can serve as a framework to compare ZFC-based and cZFC^--based set theories?

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- **Note:** The assumption "There is a run that gets every set" is equivalent to a global well-order (and hence CH when we have countabilising!).

- One last extra.
- In line with the Cohen-Scott Paradox: How much height absoluteness can we feed in?

Thanks for listening!