Countabilism: A Survey

Neil Barton
Slides available via the "Blog" section of my website
https://neilbarton.net/blog/





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- I'll be back.

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MAIN AIMS.

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MAIN AIMS.

- 1. Present some motivations for countabilism from different sources.
- 2. Identify where interesting points of contact with the group might be for future work.

Introduction

SECOND-ORDER ARITHMETIC

'Weak' systems

'RICH' NON-MODAL THEORIES

'Real' foundations

Modal Theories

CHALLENGES

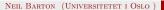
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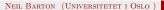
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- There's some trivial examples of this (e.g. various species of ultrafinitism).
- I want a version that at least allows the existence (again, in some form) of an infinite set.





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- This immediately puts you in the space of second-order arithmetic, and so...

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- Importantly also: Linnebo-Shapiro modal predicativism (more on this later)...

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- A small c prefix, will be used to denote the addition of Count, so cZFC− is ZFC− + Count.

■ ZFC⁻ is ZFC⁻ with the following Collection Scheme added.



■ Collection:

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- This is provably equivalent to the DC-scheme; the scheme of assertions claiming that for each formula $\phi(x,y,z)$ and parameter a, if for every x there is a y such that $\phi(x, y, a)$ holds, then there is an ω -sequence $\langle x_n | n \in \omega \rangle$ such that for all $n, \phi(x_n, x_{n+1}, a)$ holds. (i.e. If a definable relation has no terminal nodes, we can make ω -many dependent choices on its basis.)

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- **Projective Determinacy** or PD will be rendered as the schema asserting that every definable class of reals has a winning strategy.
- **Side note (but important)!** If you go to a class theory, the **Limitation of Size** principle that all classes are the same size is **equivalent** to CH!

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FACT.

(Work of Zarach, Gitman, Hamkins, Johnston, S. Friedman) cZFC- $\not\Rightarrow$ cZFC- $\not\Rightarrow$ cZFC- $\not\Rightarrow$ cZFC-

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 $\bar{\mathbb{I}}_1^1$ -PSP $\not\Rightarrow$ PD over any of the above theories (converse is immediate).

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Over cZFC⁻, PD is equiconsistent with ZFC + "Lots of Woodin cardinals" (in fact PD is equivalent to many 'nice' inner models of ZFC with Woodin cardinals).

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- For now I'll just mention some non-modal axioms.

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- Ordinal Inner Model Hypothesis (OIMH) is the EIMH restricted to ordinal parameters.

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FACT (ALSO NEAT EXERCISE).

The FSA and Count are equivalent modulo ZFC—.

FACT.

 $FSA \Rightarrow ASGA$, but they are equiconsistent (with ZFC-).

Introduction SOA 'Weak' systems Stronger theories 'Real' foundations Modal theories Challenges

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FACT.

The OIMH implies that $0\sharp$ exists.

FACT.

You need some impredicative class theory to formulate both the EIMH and OIMH.

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- Is there a different way we might motivate strong axioms?

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- An idea myself and Chris Scambler have been toying with: Take the real numbers as your main foundational object.
- These are the kinds of things that can be used to measure (possible?) spatial magnitudes.
- This reverses the usual 'foundational' arrow.

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- \blacksquare Our 'foundation' would then be SOA + PD.
- And the set theory that could thereby be interpreted would be cZFC⁻_{Ref} (I'm happy to throw Dependent Choice too, but this is a bolt-on) + PD.

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- (Linnebo-Shapiro) Modal predicativism. Clearly countabilist (at least allowing classes) but what one gets depends on a bit on whether strict or liberal. Pause for discussion.

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 - The necessitation of the axioms of ZFC.
- Imprecise point. Øystein, Sam Roberts, and myself have been playing with some axioms that just assume that there's a 'genie' that can only do 'countable work'. How does this compare? Pause for discussion.

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Adding in a separate modality for height vs. width, and asserting the Linnebo axioms for the height modality gets you mutual interpretability with cZFC-+ RCR-sauce + $\tilde{\Pi}_1^1$ -PSP (see Scambler's contribution to the *Palgrave Companion to the Philosophy of Set Theory*)

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OBSERVATION.

There may be sets floating around that aren't obtained by forcing or collapsing, but still get in to a Kripke frame for the axioms...

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- And **Reflection** is non-modally equivalent to the DC-Scheme, so...
- All of these assertions are floating around extensions of .2 or choice-like ideas...(not sure what to make of this, pause for discussion).

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- I do think that these discussions show that the notion of cardinal size is very auxiliary assumption dependent.

■ Challenge 3. How to make sense of the iterative conception and stage theory.

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CHALLENGES

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- Option I. Reject the iterative conception (we are all a bit obsessed with it).
- Option II. Take the modal theories to be giving you a version of the iterative conception (advanced in the booklet).

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- Perhaps this can serve as a framework to compare **ZFC**-based and **cZFC**-based set theories?

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- Note: The assumption "There is a run that gets every set" is equivalent to a global well-order (and hence CH when we have countabilising!).

- One last extra.
- In line with the Cohen-Scott Paradox: How much height absoluteness can we feed in?

Thanks for listening!