Countabilism: A Survey

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- I wanted to start by saying thanks to the group for a wonderful time in Oslo.
- I'll be back.

- There's been some interest in countabilism recently.
- Or at least I've been interested in it...
- I thought it might be worth just giving a quick survey of some issues.

Main Aims.

1. Present some motivations for countabilism from different sources.
2. Identify where interesting points of contact with the group might be for future work.

Introduction

Second-order arithmetic

'Weak' systems

'Rich' non-modal theories

'Real' foundations

Modal theories

Challenges

- Let's just start by getting some theoretical background on the table.
- Countabilism states, in some form:
- **Axiom.** Every set is countable.
- There's some trivial examples of this (e.g. various species of ultrafinitism).
- I want a version that at least allows the existence (again, in some form) of an infinite set.

- Once you're playing this game, we should note the following:

Observation.
Countabilism implies (given enough background assumptions) that every set is coded by a real number (one direction Dedekind cuts, the other way by coding sets as well-founded trees).

- You're thus (sort of) saying that the only difference between discussing sets and real numbers is a matter of coding.
- This seems innocent enough to me, but maybe I'm smuggling something in.
- This immediately puts you in the space of second-order arithmetic, and so...

- **First route to countabilism.** You have some sort of view of sets (or perhaps set formation) as restricted.
- If you've somehow landed on one of the 'Big Five' (or somewhere else in the SOA-Zoo), then you're a kind of countabilist. This goes for
  - Bishop-style constructive mathematics (RCA_0)
  - Hilbert-style finitistic reductionism (WKL_0).
  - Weyl-Feferman predicativism (ACA_0)
  - Friedman-Simpson predicative reductionism (ATR_0)
  - Impredicative math (above Π^1_1-Comprehension).
- Importantly also: Linnebo-Shapiro modal predicativism (more on this later)...

- Instead most of what I've worked on has been 'rich' theories of sets (in $\mathscr{L}_\in$).
- Count is the axiom asserting that every set is countable (is bijective with ω).
- ZFC− is ZFC (formulated with Replacement) and the Powerset axiom deleted.
- A small c prefix, will be used to denote the addition of Count, so cZFC− is ZFC− + Count.

- ZFC^- is $\mathsf{ZFC}-$ with the following **Collection Scheme** added.
- **Collection**:
$\forall a \forall x \in a \exists y(\phi(x,y) \rightarrow (\exists z \forall x \in a \exists y \in z \phi(x,y)))$.
- ZFC^-_{Ref} is ZFC with the following reflection scheme added.
- **Reflection.** $\forall x \exists a(x \in a \wedge$ 'a is transitive' $\wedge\, \phi \leftrightarrow \phi^a)$
- This is provably equivalent to the DC-scheme; the scheme of assertions claiming that for each formula $\phi(x,y,z)$ and parameter a, if for every x there is a y such that $\phi(x,y,a)$ holds, then there is an ω-sequence $\langle x_n | n \in \omega \rangle$ such that for all n, $\phi(x_n, x_{n+1}, a)$ holds. (i.e. If a definable relation has no terminal nodes, we can make ω-many dependent choices on its basis.)

- Other important axioms:
- The $\underset{\sim}{\Pi}^1_1$**-Perfect Set Property** ($\underset{\sim}{\Pi}^1_1$-PSP) is the claim that every Π^1_1-definable class of reals (i.e. countable sets) has the perfect set property.
- Here (by work of Solovay and Taranovsky) this can just be the schema (in $\mathscr{L}_\in$) 'For every real x, $L[x] \models \mathsf{ZFC}$'.
- **Projective Determinacy** or PD will be rendered as the schema asserting that every definable class of reals has a winning strategy.
- **Side note (but important)!** If you go to a class theory, the **Limitation of Size** principle that all classes are the same size is equivalent to CH!

Fact.

ZFC− + Count and SOA are bi-interpretable.

Fact.

(Work of Zarach, Gitman, Hamkins, Johnston, S. Friedman)
$\mathsf{cZFC}- \not\Rightarrow \mathsf{cZFC}^- \not\Rightarrow \mathsf{cZFC}^-_{Ref}$

Fact.

$\underset{\sim}{\Pi}^1_1$-PSP $\not\Rightarrow$ PD over any of the above theories (converse is immediate).

Fact

cZFC−, cZFC$^-$, and cZFC$^-_{Ref}$ are equiconsistent.

Fact.

Over cZFC$^-$, $\underset{\sim}{\Pi}^1_1$-PSP is equiconsistent with ZFC (in fact $\underset{\sim}{\Pi}^1_1$-PSP is equivalent to many 'nice' inner models of ZFC).

Fact.

Over cZFC$^-$, PD is equiconsistent with ZFC + "Lots of Woodin cardinals" (in fact PD is equivalent to many 'nice' inner models of ZFC with Woodin cardinals).

- How to motivate these theories?
- (One option we'll talk about later, modal theories...)
- For now I'll just mention some non-modal axioms.

- **Second route to countabilism.** Some sort of 'richness' assumption that implies Count.
- Some things we (myself and Sy-David Friedman) looked at.
- **Forcing Saturation Axiom.** (FSA) There is a generic for any partial order and set-sized family of dense sets.
- **Axiom of Set Generic Absoluteness.** (ASGA) Let $\phi(\vec{a})$ be a formula in the parameters $\vec{a}$. Then if there is a set-forcing extension of of V such that ϕ, then ϕ holds in V.
- **Extreme Inner Model Hypothesis.** (EIMH) Let $\phi(\vec{a})$ be as above. If there is a class-forcing extension such that $\phi(\vec{a})$ then $\vec{a}$ holds in V.
- **Ordinal Inner Model Hypothesis** (OIMH) is the EIMH restricted to ordinal parameters.

Some results:

Fact (also neat exercise).

The FSA and Count are equivalent modulo ZFC−.

Fact.

FSA $\nRightarrow$ ASGA, but they are equiconsistent (with ZFC−).

Fact.

The EIMH is inconsistent with cZFC^-_{Ref}.

Fact.

The OIMH is consistent relative to ZFC + PD.

Fact.

The OIMH implies that $0\sharp$ exists.

Fact.

You need some impredicative class theory to formulate both the EIMH and OIMH.

- I think what's emerging here is a landscape somewhat similar to the large cardinals and ZFC world.
- But it's really hard to see how to investigate the space between the OIMH and EIMH.
- (This is no different from reflection principles in a sense—see the Koellner paper on this—you go pretty quickly from second-order reflection to inconsistency.)
- Is there a different way we might motivate strong axioms?

- **Third route to countabilism.** The primacy of real numbers and determinacy?
- An idea myself and Chris Scambler have been toying with: Take the real numbers as your main foundational object.
- These are the kinds of things that can be used to measure (possible?) spatial magnitudes.
- This reverses the usual 'foundational' arrow.

- **Warning!** Half-baked ideas incoming!
- One axiom you might want to have is that the probability of hitting a point on a dartboard doesn't change if you move it around in space.
- This is basically just determinacy (about definable classes of reals)!
- This turns an objection to Freiling's darts (made in the ZFC-context) on its head.
- Our 'foundation' would then be SOA + PD.
- And the set theory that could thereby be interpreted would be cZFC^{-}_{Ref} (I'm happy to throw Dependent Choice too, but this is a bolt-on) + PD.

- **Fourth route(s) to countabilism.** I have some modal theory of sets that interprets Count (plus a suitable theory) under the potentialist translation.
- (Linnebo-Shapiro) **Modal predicativism.** Clearly countabilist (at least allowing classes) but what one gets depends on a bit on whether strict or liberal. **Pause for discussion.**

- (Scambler) Considers the following key axioms:
  - Collapse$^\Diamond$
  - **Possible Generics** For any forcing partial order and family of dense sets there could be a generic intersecting all the dense sets.
- (Me, but based on Chris' work) Considers the following:
  - **Possible Generics**
  - **Ordinal Definiteness Schema.** (Think Barcan for the ordinals) $\forall x(\text{`}x \text{ is an ordinal'} \rightarrow \Box\phi(x)) \rightarrow \Box\forall y(\text{`}y \text{ is an ordinal'} \rightarrow \phi(y))$
  - The necessitation of the axioms of ZFC.
- **Imprecise point.** Øystein, Sam Roberts, and myself have been playing with some axioms that just assume that there's a 'genie' that can only do 'countable work'. How does this compare? **Pause for discussion.**

Fact.

Scambler's (new!) system interprets cZFC− + whatever Replacement/Collection/Reflection (henceforth RCR-sauce) you're willing to stick in, and is mutually interpretable with cZFC− + RCR-sauce.

Fact.

Adding in a separate modality for height vs. width, and asserting the Linnebo axioms for the height modality gets you mutual interpretability with cZFC− + RCR-sauce + $\undertilde{\Pi}^1_1$-PSP (see Scambler's contribution to the *Palgrave Companion to the Philosophy of Set Theory*) available in all good bookshops in 2024).

Fact.

My system also interprets cZFC− + RCR-sauce + $\underset{\sim}{\Pi}^1_1$-PSP.

Observation.

There may be sets floating around that aren't obtained by forcing or collapsing, but still get in to a Kripke frame for the axioms...

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- Some remarks about these modal axioms (in particular, how spicy is the RCR-sauce?)
- **Replacement** functions a bit like a 'super .2' axiom or $\omega.2$ if you will.
- In this respect, it's similar to the 'jump' in the Linnebo-Shapiro potentialist setting.
- **Collection** is a bit like a kind of choice-ish principle.
- **Reflection** is also a kind of extension of .2, depending on the setting.
- And **Reflection** is non-modally equivalent to the DC-Scheme, so...
- All of these assertions are floating around extensions of .2 or choice-like ideas...(not sure what to make of this, **pause for discussion**).

- **Challenge 1.** How to interpret 'standard' math at the level of third-order analysis? (Discussed in the booklet 'Iterative Conceptions of Set' a bunch.)
- **Challenge 2.** Is this a restrictive perspective?
- Toby Meadows has a result ('What is a restrictive theory') that variants of cZFC− are restrictive, I have a result ('Is (un)countabilism restrictive?') that argues that ZFC is restrictive.
- I'm not hopeful of making progress (beyond the isolation of the different interesting 'restrictiveness' notions).
- I do think that these discussions show that the notion of cardinal size is very auxiliary assumption dependent.

- **Challenge 3.** How to make sense of the iterative conception and stage theory.
- **Option I.** Reject the iterative conception (we are all a bit obsessed with it).
- **Option II.** Take the modal theories to be giving you a version of the iterative conception (advanced in the booklet).

- **Option III.** Devise a stage theory that is nice and countabilist.
- Ethan Brauer has proposed (see the *Palgrave Companion*) a forcing potentialist conception on which:
  - We add to a choice sequence.
  - Close under definability.
  - Take unions at limits.
- It seems to me that a possible bolt-on is the assumption that every set can be countablised at some point of the process.
- Perhaps this can serve as a framework to compare ZFC-based and cZFC^--based set theories?

- However the addition of definable powersets, whilst natural, seems unnecessary...
- I think a single wand that forms an arbitrary set from what you have is enough.
- **Note:** The assumption "There is a run that gets every set" is equivalent to a global well-order (and hence CH when we have countabilising!).

- One last extra.
- In line with the Cohen-Scott Paradox: How much height absoluteness can we feed in?

Thanks for listening!