More Imperatival Foundations: What to make of it all?

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Forskningsrådet

- You've all seen the stuff on postulationism a few times now.
- You can find the draft 'Make It So: Imperatival Foundations for Mathematics' on the PhilArchive.
- As always: Big up Ethan Russo and Chris Scambler, they really drove the project forward.
- In this presentation I want to facilitate discussion about some of the main issues surrounding postulationism and get your thoughts.
- I'll try to cut down on details and me talking.

Main Question

How different are declarative and imperatival foundations? In particular:

1. Are there different philosophical positions suggested by the two?
2. What are the relationships between them?
3. Is one better than the other?
4. Are there open challenges it would be good for the imperativalist to address?

Introduction

What is postulationism? (with intro 20 min)

Postulationist responses to the paradoxes (20 min)

Issues of consistency (20 min)

Challenges: Postulataionism (huh?) what is it good for? (20 min/if there's time)

Thanks for listening!

- Previously I've bombarded you with type theory.
- Rather than giving you all the details (which you can get anyway in 'Make It So'), I want to try and isolate what's common to imperatival approaches.
- **Rough** distinction:
- **Declarative foundations:** We describe some (platonically?) existing objects which can be used as mathematical foundations (i.e. what objects exist).
- **Imperatival foundations:** We describe some imperatives that can be used as mathematical foundations (i.e. what can be done.)

- **The Standard View:** The imperatival idiom should (in principle) be translated away in favour of the declarative one.
- But note:
  1. Different philosophical approaches are interesting.
  2. Mathematics is replete with the use of imperatival terms.
- To sharpen what we need from imperatival foundations, let's go back to the **OG of imperatival foundations**: Euclid.[1]

[1] At least under one interpretation.

- In Euclid, we see a distinction between two sorts of proposition problemata and theoremata.
- **Theoremata** concern results that show some relation between geometric objects must obtain
- **Example.** (Proposition 4) shows that two triangles with two similar sides and the same angles are similar.
- **Problemata** which detail something approximating a recipe for the construction of a particular geometrical figure.
- **Example.** (Proposition 1) shows how to construct an equilateral triangle about a given segment.

- Let's dissect what we need to carry out such an approach. We need:
- **Basic imperatives.** Some stock of basic imperatives.
- In our case "Make an F!", which will make a unique object with the property of being F.
- **Note:** The choice of these is suprisingly non-obvious in certain cases, getting us into Zeno-type worries (e.g. Do I have to construct an arc before I construct a circle?).
- **Compositional rules.** How to build new imperatives from old.
    1. $i; j$ "Do i; then, do j!"
    2. $p \rightarrow i$ "If p, then do i!"
    3. $\forall x i(x)$ "Do i to every x!"
    4. i^* "Do i forever!"
- **Note:** We define i^* via the universal, Fine takes it as primitive.
- **Note:** $(p \rightarrow i)^*$ has the force of "While p, do i!"

- Some way of:
  1. Constraining what imperatives do and talking about what holds after an imperative has been done.
  2. Saying when an imperative is doable/executable.
- We may as well take this to be some form of modal logic, with modal operators $[i]$ for each imperative as well as a general $\Box$.
- **Note:** We take i^* to be executable when you reach a fixed point, so we don't allow non-terminating i^* (ones that always get you new things) to be executable.

- **Example.** Arithmetic.
- Two imperatives:
  1. ζ, intended as "Make a number!", (which in the wash comes out as "Make zero!")
  2. σ, intended as "Make the successor of everything you have!"
  3. In order to capture this idea, we lay down constraints on how the imperatives behave.
- **Successor Uniqueness** $(Sxy \wedge Sxz) \rightarrow y = z$
- **Predecessor Uniqueness** $(Syx \wedge Szx) \rightarrow y = z$
- **Number Stability** $Nx \rightarrow \Box Nx$
- **Successors are Numbers** $Sxy \rightarrow (Nx \wedge Ny)$
- **Predecessor Stability** $Sxy \rightarrow \Box Sxy$
- **Predecessor Inextensibility** $\forall y(Syx \rightarrow \Box Fy) \rightarrow \Box \forall y(Syx \rightarrow Fy))$

Fact.

Given these constraints, we can then show that, starting from scratch, and given a background logic of imperatives, that after $Num =_{df} \zeta; \sigma^*$ ("Introduce zero; then introduce successors forever!") has been done, PA_2 holds.

- ρ is the imperative "For every plurality X, introduce a set coextensive with X!", constrained by:
- **Extensionality** $\forall x, y(Set(x, X) \wedge Set(y, X)) \rightarrow x = y)$
- **(Plurality-Uniqueness)** $\forall X, Y(Set(x, X) \wedge Set(x, Y) \rightarrow X = Y)$
- **Set Stability** $Sx \rightarrow \Box Sx$
- **Set-member** $x \in y \rightarrow Sy$
- **Membership Stability** $x \in y \rightarrow \Box x \in y$
- **Membership Inextensibility** $\forall y(y \in x \rightarrow \Box Fy) \rightarrow \Box \forall y(y \in x \rightarrow Fy)$

Fact.

Starting from scratch, the imperative "Whilst the ordinals are accessible, do ρ!" will result in ZFC_2 being true of the sets.

Question.

Does the distinction between imperatival and declarative foundations come down to merely the use of modal logic/ideas?

Question.

For example, should we think of **any** modal set theory as a species of imperatival foundation?

- I want to start some discussion on two interlinked issues **paradoxes** and **consistency**.
- Let ρ be as before, the command to take each plurality xx and make a set with those things as its elements.
- We can now point out:

FACT

(Generative Russell) “Do powerset forever!” is never executable. (That’s why we needed the ‘hedge’ that the ordinals are accessible.)

- Fine takes this to show that, roughly speaking, the sets are indefinitely extensible.
- This is because he thinks that ρ is necessarily executable.
- But why think that? (I grant that it's natural.)

Say that a postulate is conservative if it is necessarily consistent..., i.e. consistent no matter what the domain. Then it should be clear that the simple postulate ZERO is conservative and that the simple postulate [σ] is conservative... For these postulates introduce a single new object into the domain that is evidently related in a consistent manner to the preexisting objects. (Fine, OKOMO preprint)

- Fine thinks that ρ is also conservative (i.e. necessarily executable).
- There's certainly a red sensible intuition here (I'm just introducing a set of everything I've got).
- But if you've strong generality absolutist intuitions, ρ is not necessarily executable as a matter of logic.
- It's not "Consistent no matter what the domain"
- For a universe imperativalist there's a perfectly good (!?) imperative that produces the universist domain "While ρ is executable, do ρ!"

Question.

Do imperatival foundations really suggest the indefinite extensibility approach? Or is it rather that there's just the same multiple ways to go as in the declarative case, and different people with different intuitions are likely to go the correspondingly different ways when they think about imperatives?

- **Suggestion.** Every consistently executable (**Note:** Saying what that means isn't so easy) imperative is necessarily executable.
- **Problem:** there are incompatible imperatives.
- Let ϵ be the imperative to enumerate the domain (I'd need to lay down postulational constraints on how this works, but let's just go with it).

Imperatival Cohen-Scott Paradox

It is not possible for both ρ and ϵ to be necessarily executable, without subsequent modifications to the logic. (**Note:** This is really just an idea, not properly formalised yet.)

- Is this just the countabilist vs. uncountabilist debate incarnated in imperatival logic?
- I suspect that Roberts-style inconsistency results will be formalisable in imperatival terms.
- In fact there's a way of seeing the Generative Russell as inconsistent imperatives.
- Both ρ and $(\langle \rho \rangle \top \rightarrow \rho)^*$ could be necessarily executable, depending on background assumptions.
- But they are not jointly so, by the Generative Russell.

Question.

Other examples of this kind? It looks like things like parities ("Introduce a parity for what you have!" vs. "Make an infinite set!") have similarities, but seem more "cooked up".

Question.

Are there links to abstraction principles emerging here?

Question.

What about imperatival Russell-Myhill?

- Closely related to paradox are ideas of consistency.
- How do we know when a theory is consistent?
- We want to transform this into knowing when some imperatives are executable.
- There's a natural thought (apologies for the folks who've heard me talk about this a bunch) consistency should flow upward.
- In our setting, the consistency of "Do i, then do j!" follows from the necessary executability of i and j.
- But this alone won't get you the executability of Num.

It should also be clear that each of the operations for forming complex postulates will preserve conservativity. For example, if i and j are conservative then so is $i; j$, for i will be executable on any given domain and, whatever domain it thereby induces, will be one upon which j is executable. But it then follows that NUMBER is consistent, as is any other postulate that is formed from conservative simple postulates by means of the operations for forming complex postulates. (Consistency can also be demonstrated for the higher reaches of set theory, though not in such a straightforward way). (Fine, OKOMO preprint)

- The following idea is natural (what we've previously called Universal Upwards Transmission or UUT): "Every i is doable iff the command to do them *all* is doable".
- **Formally:** $\langle \forall x I x \rangle \top \leftrightarrow \forall x \langle I x \rangle \top$
- This leads immediately (assuming the necessary executability of ρ) into the Generative Russell.
- But without the necessary excitability of ρ is not obviously inconsistent.

Question.

Does UUT lead to other paradoxes? (e.g. Russell-Myhill?)

Question.

Are there restricted versions of **UT** or other ways of achieving the 'Goldilocks zone' of upwards transmission? (Brute restriction to iteration 'lengths' (e.g. ω-sequences, κ-sequences, etc.) seems to be one route, but doesn't seem very attractive.)

Question.

However much I press on UUT, I feel there is something to it:

- We all believe that PA_2 (or pick PRA if you like, or whatever) is consistent. Why?
- I think often the reasoning goes something like this; we can imagine or conceive of a structure on which the axioms are true.
- But do we just intuit an ω-sequence directly?
- I don't think so: We imagine that we can always add successors (this seems very plausible).
- What could stop us bundling them together?
- But this invites in the question: **Where is the line?**
- A kind of "yes to all non-obviously bad imperatives" is tempting. But as we've seen that's enough to get us into trouble.

Observation.
Our system is, at least in comparison to standard set theory, **clunky**.

- I do think it's worthwhile (e.g. for formalising the intuitions a mentioned).
- But to really earn its keep, I think it needs to solve philosophical problems or produce good mathematics.
- As I've said, I think its significance for the "usual" problems of consistency and potentialism is a little unclear.
- It would be good to look for other applications.

- **Problem 1.** Both we and Fine assume that there's only one way to execute an imperative.
- We use axiom **D** $[i]\phi \leftrightarrow \langle i \rangle \phi$.
- Fine seems to see this more as at the heart of the matter—an imperative should give you a categorical specification of a domain—whereas for us it's more a matter of convenience—zeroing in on a class of imperatives that we take to be especially interesting or important.
- But there's loads of indeterminate constructions.
- **Example.** "Extend a given line segment!"
- **Example.** "Build a triangle!"
- **Example.** "Add a forcing generic!"
- **Example.** "Build a group, then permute it!"

- I don't see any obstacle to just dropping the **D**-axiom, it can always be reintroduced.
- In fact it's a nice way of axiomatising the idea that an imperative results in a unique (up to isomorphism) domain.
- A categorical imperative, if you will (sorry).
- Getting rid of **D** comes with some challenges: When proving necessitated universals, we regularly go from $[i]\phi \rightarrow [i]\psi$ to $[i](\phi \rightarrow \psi)$ (towards $[i]\forall x(F(x) \rightarrow G(x))$ more generally).
- But I think if this can be made to work, there's some reason to be hopeful about possible the following philosophical payoff.

- There's a puzzle that people get into in the arbitrary reference literature.
- Suppose I take an arbitrary triangle T.
- **Question.** Is T isosceles?
- **Option 1.** (Magidor) We should think of reference to T as naming a particular triangle via arbitrary reference, we just don't know which.
- **Disadvantage.** What mysterious faculty enables this arbitrary reference?
- **Option 2.** (Horsten, Fine) There are things called 'arbitrary objects', and T refers to an (the?) arbitrary triangle.
- **Disadvantage.** These seem like creatures of the night to me (I don't have much better to say, change my mind).

- An indeterminate imperative might avoid these puzzles.
- "Construct a triangle!" **Question.** Is it isosceles?
- Well, on the one hand, once you have built a triangle, since every triangle is either isosceles or not, it is one of the two.
- But (perhaps using backtracking operators) we'd also have $\downarrow \Diamond$"Isosceles" and $\downarrow \Diamond$"Non-isosceles".
- It's thus not necessary when you haven't built anything that what you'll get will (not) be isosceles, and you couldn't have assumed that it would be, even though when we have built it, it is (not) isosceles.
- This is no (?!) more mysterious than the fact that if I ask you to build me a table, it might be three-legged, or it might be non-three-legged, but whatever I get will be either three-legged or non-three-legged.

- An additional complication is that we assume **Economy** principles, that no more than necessary is done to execute a command.
- These are important (e.g. you can get failures of induction if you introduce an unconnected $\mathbb{Z}$-chain as well as introducing successors).
- We might think there's a problem here with indeterminate operators.
- e.g. Couldn't I have always made a smaller line?

- But the way it's natural to formalise Economy is compatible with indeterminacy (it basically says that when you make a new ϕ, you just get one).
- **Example.** "Extend l!" is different from "Extend l; then, close under the formation of sub-lines!".
- **Example.** "Add in a generic G for $\mathbb{P}$!" is different from "Add in a generic G for $\mathbb{P}$; then, close under the evaluation of $\mathbb{P}$-names!"
- I do think that a notion of imperative has to be well-founded.
- So perhaps a substitute for **D** might be that all imperatival modalities are well-founded?
- Should be formalisable (perhaps with more expressive resources) but not quite sure of the best way yet.

Very general question.

What do y'all think of indeterminate imperatives? Are they legit?

Question.

I don't think postulationism will ever be non-clunky for set theory or arithmetic. But there areas of mathematics where postulationism would be better?

- What about arithmetic simpliciter?
- What about recursion theory and computational complexity?
- Approaches to predicativism?
- Motivations for other kinds of axioms (e.g. large cardinals)?

- Possibility for studying interesting axioms in the context of set theory.
- e.g. Indefinite iteration seems made for discussing fixed points.
- For example, we've got our Set_h that builds the least inaccessible fixed point, and $\rho; Set_h$ gets you the same (starting from scratch).
- But $\rho; Set_h; \rho; Set_h$ will get us the next.
- So finite iterations of $\rho; Set_h$ get us more.
- Now what about $i =$"Do all finite iterations of Set_h!"? (This we can just steal from the ancestral and the arithmetic case.)
- Now consider $\rho; i$...
- Now what about: "Whilst the class of inaccessibles is accessible, do ρ"
- Now it seems we've got a hyperinaccessible...

- Let's now consider: **Axiom F** Every normal function on the ordinals has a regular fixed point.
- Effectively this works out as the second-order axiom "Ord is Mahlo".
- We can formulate the imperative: "Whilst there is a normal function F on the ordinals with no regular fixed-point, do ρ!"
- It would be great to figure out how to do this in terms of applications of the Fs, rather than just the defining condition.

- Contrast: “Whilst there’s no measurable cardinal, do ρ!”
- It would be more interesting if there were imperatives of the form “Do F forever; then do J forever...”.
- Perhaps: “For every F, if F doesn’t have a regular fixed-point, do F forever!” will do?
- The resulting domain will then (by assumption) have a regular fixed point for every F.
- I am very suspicious that this will all link nicely into the standard story concerning indescribables, reflection, and fixed points on normal functions.
- But probably let’s walk before we run...

Question.

What about other areas that have nice with fixed point characterisations?

- Kripke-style truth.
- Determinacy operators and truth (Welch, Scambler).
- Recursion theory?
- Computational complexity?

Thanks for listening!