# More Imperatival Foundations: What to make of it all?

Neil Barton (jww Ethan Russo and Chris Scambler)

Slides available via the "Blog" section of my website
https://neilbarton.net/blog/

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Forskningsrådet

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- I'll try to cut down on details and me talking.

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3. Is one better than the other?
4. Are there open challenges it would be good for the imperativalist to address?

Introduction

What is postulationism? (with intro 20 min)

Postulationist responses to the paradoxes (20 min)

Issues of consistency (20 min)

Challenges: Postulataionism (huh?) what is it good for? (20 min/if there's time)

Thanks for listening!

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- **Rough** distinction:
- **Declarative foundations:** We describe some (platonically?) existing objects which can be used as mathematical foundations (i.e. what objects exist).
- **Imperatival foundations:** We describe some imperatives that can be used as mathematical foundations (i.e. what can be done.)

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- **The Standard View:** The imperatival idiom should (in principle) be translated away in favour of the declarative one.
- But note:
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  2. Mathematics is replete with the use of imperatival terms.
- To sharpen what we need from imperatival foundations, let's go back to the **OG of imperatival foundations**: Euclid.[1]

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- **Note:** $(p \to i)^*$ has the force of "While $p$, do $i$!"

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- **Note:** We take $i^*$ to be executable when you reach a fixed point, so we don't allow non-terminating $i^*$ (ones that always get you new things) to be executable.

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- **Successor Uniqueness** $(Sxy \wedge Sxz) \rightarrow y = z$
- **Predecessor Uniqueness** $(Syx \wedge Szx) \rightarrow y = z$
- **Number Stability** $Nx \rightarrow \Box Nx$
- **Successors are Numbers** $Sxy \rightarrow (Nx \wedge Ny)$
- **Predecessor Stability** $Sxy \rightarrow \Box Sxy$
- **Predecessor Inextensibility** $\forall y(Syx \rightarrow \Box Fy) \rightarrow \Box \forall y(Syx \rightarrow Fy))$

**Fact.**

Given these constraints, we can then show that, starting from scratch, and given a background logic of imperatives, that after $Num =_{df} \zeta ; \sigma^*$ ("Introduce zero; then introduce successors forever!") has been done, $\mathsf{PA}_2$ holds.

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- **(Plurality-Uniqueness)** $\forall X, Y(Set(x, X) \wedge Set(x, Y) \rightarrow X = Y)$
- **Set Stability** $Sx \rightarrow \Box Sx$
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**Fact.**

Starting from scratch, the imperative "Whilst the ordinals are accessible, do $\rho$!" will result in $\mathsf{ZFC}_2$ being true of the sets.

**Question.**

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For example, should we think of **any** modal set theory as a species of imperatival foundation?

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**FACT**

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*Say that a postulate is conservative if it is necessarily consistent..., i.e. consistent no matter what the domain. Then it should be clear that the simple postulate ZERO is conservative and that the simple postulate [$\sigma$] is conservative... For these postulates introduce a single new object into the domain that is evidently related in a consistent manner to the preexisting objects. (Fine, OKOMO preprint)*

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- It's not "Consistent no matter what the domain"
- For a universe imperativalist there's a perfectly good (!?) imperative that produces the universist domain "While $\rho$ is executable, do $\rho$!"

**Question.**

Do imperatival foundations really suggest the indefinite extensibility approach? Or is it rather that there's just the same multiple ways to go as in the declarative case, and different people with different intuitions are likely to go the correspondingly different ways when they think about imperatives?

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**Imperatival Cohen-Scott Paradox**

It is not possible for both $\rho$ and $\epsilon$ to be necessarily executable, without subsequent modifications to the logic. (**Note:** This is really just an idea, not properly formalised yet.)

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- But they are not jointly so, by the Generative Russell.

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What about imperatival Russell-Myhill?

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- In our setting, the consistency of "Do $i$, then do $j$!" follows from the necessary executability of $i$ and $j$.
- But this alone won't get you the executability of $Num$.

*It should also be clear that each of the operations for forming complex postulates will preserve conservativity. For example, if $i$ and $j$ are conservative then so is $i; j$, for $i$ will be executable on any given domain and, whatever domain it thereby induces, will be one upon which $j$ is executable. But it then follows that NUMBER is consistent, as is any other postulate that is formed from conservative simple postulates by means of the operations for forming complex postulates. (Consistency can also be demonstrated for the higher reaches of set theory, though not in such a straightforward way). (Fine, OKOMO preprint)*

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- But without the necessary excitability of $\rho$ is not obviously inconsistent.

**Question.**

Does UUT lead to other paradoxes? (e.g. Russell-Myhill?)

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Are there restricted versions of **UT** or other ways of achieving the 'Goldilocks zone' of upwards transmission? (Brute restriction to iteration 'lengths' (e.g. $\omega$-sequences, $\kappa$-sequences, etc.) seems to be one route, but doesn't seem very attractive.)

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- I don't think so: We imagine that we can always add successors (this seems very plausible).

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- But this invites in the question: **Where is the line?**
- A kind of "yes to all non-obviously bad imperatives" is tempting. But as we've seen that's enough to get us into trouble.

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- It would be good to look for other applications.

- **Problem 1.** Both we and Fine assume that there's only one way to execute an imperative.

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- Getting rid of **D** comes with some challenges: When proving necessitated universals, we regularly go from $[i]\phi \rightarrow [i]\psi$ to $[i](\phi \rightarrow \psi)$ (towards $[i]\forall x(F(x) \rightarrow G(x))$ more generally).

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- But I think if this can be made to work, there's some reason to be hopeful about possible the following philosophical payoff.

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- **Disadvantage.** These seem like creatures of the night to me (I don't have much better to say, change my mind).

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- It's thus not necessary when you haven't built anything that what you'll get will (not) be isosceles, and you couldn't have assumed that it would be, even though when we have built it, it is (not) isosceles.
- This is no (?!) more mysterious than the fact that if I ask you to build me a table, it might be three-legged, or it might be non-three-legged, but whatever I get will be either three-legged or non-three-legged.

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- e.g. Couldn't I have always made a smaller line?

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- So perhaps a substitute for **D** might be that all imperatival modalities are well-founded?
- Should be formalisable (perhaps with more expressive resources) but not quite sure of the best way yet.

**Very general question.**

What do y'all think of indeterminate imperatives? Are they legit?

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- What about recursion theory and computational complexity?
- Approaches to predicativism?
- Motivations for other kinds of axioms (e.g. large cardinals)?

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- Now it seems we've got a hyperinaccessible...

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- It would be great to figure out how to do this in terms of applications of the $F$s, rather than just the defining condition.

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- But probably let's walk before we run...

**Question.**

What about other areas that have nice with fixed point characterisations?

- Kripke-style truth.
- Determinacy operators and truth (Welch, Scambler).
- Recursion theory?
- Computational complexity?

Thanks for listening!