More Imperatival Foundations: What to make of it all?

Neil Barton (jww Ethan Russo and Chris Scambler)

Slides available via the "Blog" section of my website

https://neilbarton.net/blog/

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Forskningsrådet

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- What if we viewed math as less about what objects there are and more about what can be done?

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  - Kit Fine's work on 'procedural postulationism' (which in no small part inspired the present work).

Objectives.

1. Introduce the system from 'Make It So: Imperatival Foundations for Mathematics' (with Ethan Russo and Chris Scambler).

Introduction
The system and results
How different are imperatival foundations?
Model Theory
'Euclidean' mathematics
Indeterminate constructions
Down below
Up above

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1. Introduce the system from 'Make It So: Imperatival Foundations for Mathematics' (with Ethan Russo and Chris Scambler).
2. Raise some directions for future research, ones that I hope will be relevant to CFORS and interesting for the group!

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- The logic we will employ is a higher-order logic, based on a modification of the standard (functional) typed λ calculus.
- In any such typed system, one first defines a class of grammatical types, designed to represent idealised grammatical categories familiar from natural language, and then provides a vocabulary of terms in the relevant type system.
- e is a basic type, the type of entity denoting expression;
- t is a basic type, the type of truth-evaluable expression;
- (*) ι is a basic type, the type of imperatival expression;
- (**) whenever τ is a type, so too is $\tau\tau$, the type of pluralities at type τ.
- whenever σ, τ are types $\neq e$, $\sigma \to \tau$ is the type of expression which, on completion by one of type σ, yields one of type τ.

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- **Quantificational commands:** $\forall_i^\sigma$ is an operator that takes in an imperatival property and yields the quantified command to do the command to each thing.
- **Modal terms:** These connect the imperatival and declarative parts of the language. Given an imperative i, $[\cdot]$ gives us a modal operator $[i]$ (so $[i]p$ is "No matter how you do i, p" or "Let i have been done, then p!").

The axioms for the "declarative" part of our logic, are more or less standard: we assume classical propositional logic, positive free logic for the quantifiers, and other principles governing application and λ-abstraction:

PL	Every closed classical tautology
QL	Rules for positive free quantifier logic
Ex1	Existence for all the propositional and imperatival connectives and quantifiers, and identity
Ex2	Closure of existence under function application
Con	α, β and η conversion rules

We also assume a strong form of the axiom of choice as part of our higher-order logic, along with some standard principles governing the plural terms:

PlurExt	$\Box\forall x(Xx \equiv Yx) \supset X = Y$
PlurComp	$\exists X \forall x(Xx \equiv \Phi)$, no free X in Φ
Choice	$\exists f^{(\sigma\to t)\to\sigma} \forall F^{\sigma\to t}(\exists x Fx \supset F(fF))$

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- **All1.** $\exists x[i]Ey \supset [\forall x\, i(x)]Ey$
- **Det.** $\langle i\rangle p \supset [i]p$ (there's only ever one way to execute an imperative).

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- **Note:** $(p \rightarrow i)^*$ has the force of "Whilst p, do i!"

Postulational constraints. Provide "rules for construction" when we introduce a new predicate into the language, e.g. arithmetic:

Successor Uniqueness	$Sxy \wedge Sxz \supset y = z$
Predecessor Uniqueness	$Syx \wedge Szx \supset y = z$
Number Stability	$Nx \supset \Box Nx$
Successors are Numbers	$Sxy \supset Nx \wedge Ny$
Predecessor Stability	$Sxy \supset \Box Sxy$
Predecessor Inextensibility	$\forall y(Syx \supset \Box Fy) \supset \Box\forall y(Syx \supset Fy))$

Executability.

Definition.

We say that an imperative i is *executable* iff $\langle i \rangle \top$.

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If, starting from scratch, the command, "Whilst the von Neumann ordinals are accessible, do powerset!" is executable, then ZFC_2 holds of the sets.

- In the paper, we take a look at two upshots that people have thought imperatival foundations might have namely **Consistency** and **Paradoxes**.
- I'm going to suppress these (take a look at the paper).
- The two are somewhat linked, and can be traced to the following principle:
- **Universal Upwards Transmission.**

$$\langle \forall x I x \rangle \top \equiv \forall x \langle I x \rangle \top$$

Fact

(Generative Russell) "Do powerset forever!" is never executable.

Objective

Examine the similarities and differences between declarative and imperatival foundations, especially with respect to paradoxes.

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Objective.

Develop a model theory, and prove that imperatival foundations + executability of certain commands is consistent relative to standard systems of mathematics (e.g. set theory plus large cardinals).

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Objective.

In the manner of Euclid's elements, develop an approach to arithmetic and set theory on which obtaining specific sets can be construed imperatively. e.g. How to construct, given any two numbers x and y, their sum? For any two sets x and y, how to construct their union? How to build the ultrapower of a model $M \models \mathsf{ZFC}$ an M-κ-complete ultrafilter U?.

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- e.g. 3. "Build a group, then permute it!"

Objective.

Examine imperatival foundations that drops **Det** and develop an account of indeterminate constructions within it (with a view to various indeterminate mathematical constructions).

- My (limited!) understanding of constructive/intuitionistic mathematics is that it's mostly couched in the declarative idiom, with the logic modified.
- But of course constructivism and intuitionism are closely linked to computational/constructive ideas (e.g. BHK interpretation).
- There's also the question of how this all relates to the theory of computation and recursion theory (e.g. ordinal time Turing machines).

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Examine how imperatives can be used to construct large cardinals "from below".

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- You can provide a foundation for mathematics on the basis of ideas of imperatives and executability.
- I'm tempted by the idea that they're just an "equivalent way of stating the same facts".
- But the approach also seems somewhat distinctive, and there seem to be lots of interesting open questions, and it provides a way of formally systematising imperatival talk in mathematics.