More Imperatival Foundations: What to make of it all?

Neil Barton (jww Ethan Russo and Chris Scambler)
Slides available via the "Blog" section of my website
https://neilbarton.net/blog/

UNIVERSITETET
I OSLO

Forskningsrådet

- Remember the good old days before you learnt mathematical logic?
- You would happily add numbers, colour maps, and draw diagrams.
- Mathematical logic is beautiful, but can be austere.
- What if we viewed math as less about what objects there are and more about what can be done?

- The idea that mathematical foundations are to be couched in this purely declarative idiom is relatively mainstream.
- **Observation.** Certainly Euclid didn't think this way!
- Euclid's problemata are more imperitival in flavour (e.g. construct an equilateral triangle over a particular finite line segment).
- **Observation.** Nor do we (really) work solely in the declarative idiom.
  - Silvia De Toffoli's work on diagrammatic proof.
  - Fenner Tanswell, Matthew Inglis, Keith Weber on the recipe model of proof.
  - Kit Fine's work on 'procedural postulationism' (which in no small part inspired the present work).

Objectives.

1. Introduce the system from 'Make It So: Imperatival Foundations for Mathematics' (with Ethan Russo and Chris Scambler).
2. Raise some directions for future research, ones that I hope will be relevant to CFORS and interesting for the group!

Declaration

Credit should be given to Ethan Russo and Chris Scambler for developing this system! I do not speak fluent higher-order-ese I'm probably like A2 or something...

- The logic we will employ is a higher-order logic, based on a modification of the standard (functional) typed λ calculus.
- In any such typed system, one first defines a class of grammatical types, designed to represent idealised grammatical categories familiar from natural language, and then provides a vocabulary of terms in the relevant type system.
- e is a basic type, the type of entity denoting expression;
- t is a basic type, the type of truth-evaluable expression;
- (*) ι is a basic type, the type of imperatival expression;
- (**) whenever τ is a type, so too is $\tau\tau$, the type of pluralities at type τ.
- whenever σ, τ are types $\neq e$, $\sigma \to \tau$ is the type of expression which, on completion by one of type σ, yields one of type τ.

- We will also need **complex imperatival operators**.
- These allow us to make more complex commands out of our basic ones.
- For example, given commands i and j we will have the command to do i and then do j, or $i; j$.
- $p \rightarrow i$ is the command to check and see if p is true, and if it is do do i (and otherwise, do nothing).
- **Quantificational commands:** $\forall_i^\sigma$ is an operator that takes in an imperatival property and yields the quantified command to do the command to each thing.
- **Modal terms:** These connect the imperatival and declarative parts of the language. Given an imperative i, $[\cdot]$ gives us a modal operator $[i]$ (so $[i]p$ is "No matter how you do i, p" or "Let i have been done, then p!").

The axioms for the "declarative" part of our logic, are more or less standard: we assume classical propositional logic, positive free logic for the quantifiers, and other principles governing application and λ-abstraction:

- **PL** Every closed classical tautology
- **QL** Rules for positive free quantifier logic
- **Ex1** Existence for all the propositional and imperatival connectives and quantifiers, and identity
- **Ex2** Closure of existence under function application
- **Con** α, β and η conversion rules

We also assume a strong form of the axiom of choice as part of our higher-order logic, along with some standard principles governing the plural terms:

- **PlurExt** $\Box\forall x(Xx \equiv Yx) \supset X = Y$
- **PlurComp** $\exists X\forall x(Xx \equiv \Phi)$, no free X in Φ
- **Choice** $\exists f^{(\sigma\to t)\to\sigma}\forall F^{\sigma\to t}(\exists x Fx \supset F(fF))$

Axioms for imperatives: These are generally pretty intuitive. I won't go into too much detail, but I'll focus on some interesting ones.

- **Make.** $[!F]\exists F$ (No matter how you make an F, you'll have an F!)
- **Eco.** We don't do anything more than required. (See paper for formal details.)
- **Then.** $[i;j]\varphi \equiv [i][j]\varphi$.
- **All1.** $\exists x[i]Ey \supset [\forall x\, i(x)]Ey$
- **Det.** $\langle i\rangle p \supset [i]p$ (there's only ever one way to execute an imperative).

Defining indefinite iteration:

- In the system, we can define indefinite iteration.
- Roughly speaking We can define what it is to i no times (via the "do nothing" imperative).
- Then inductively you can specify what it is to be a (possibly transfinite) iteration of i (i.e. something of the form $i; i; i; i$...
- i^* is then the command to execute every iteration of i.
- Executing i^* entails reaching a fixed point.
- **Note:** $(p \rightarrow i)^*$ has the force of "Whilst p, do i!"

Postulational constraints. Provide "rules for construction" when we introduce a new predicate into the language, e.g. arithmetic:

Successor Uniqueness	$Sxy \wedge Sxz \supset y = z$
Predecessor Uniqueness	$Syx \wedge Szx \supset y = z$
Number Stability	$Nx \supset \Box Nx$
Successors are Numbers	$Sxy \supset Nx \wedge Ny$
Predecessor Stability	$Sxy \supset \Box Sxy$
Predecessor Inextensibility	$\forall y(Syx \supset \Box Fy) \supset \Box \forall y(Syx \supset Fy))$

Executability.

Definition.

We say that an imperative i is *executable* iff $\langle i \rangle \top$.

Two theorems.

Theorem.

If, starting from scratch, you can execute the command, "Introduce a number (zero)! Then, introduce successors forever!", then PA_2 holds of the natural numbers.

Theorem.

If, starting from scratch, the command, "Whilst the von Neumann ordinals are accessible, do powerset!" is executable, then ZFC_2 holds of the sets.

- In the paper, we take a look at two upshots that people have thought imperatival foundations might have namely **Consistency** and **Paradoxes**.
- I'm going to suppress these (take a look at the paper).
- The two are somewhat linked, and can be traced to the following principle:
- **Universal Upwards Transmission.**

$$\langle \forall x I x \rangle \top \equiv \forall x \langle I x \rangle \top$$

FACT

(Generative Russell) "Do powerset forever!" is never executable.

Objective

Examine the similarities and differences between declarative and imperatival foundations, especially with respect to paradoxes.

- We don't yet have a model theory for our system.
- But there's ideas in the pipeline.
- Have multi-modal Kripke frames, and interpret imperatives as accessibility relations.

Objective.

Develop a model theory, and prove that imperatival foundations + executability of certain commands is consistent relative to standard systems of mathematics (e.g. set theory plus large cardinals).

- We constructed models of theories.
- This is a little different from the Euclidean approach, there we try to find general methods for constructing an object given certain kinds of other object.
- Our approach has been to do Euclidean metamathematics has much as it has been Euclidean mathematics.
- e.g. How to construct an equilateral triangle over a given line segment? How to construct a line that cuts two rectangles into two parts of equal area?

Objective.

In the manner of Euclid's elements, develop an approach to arithmetic and set theory on which obtaining specific sets can be construed imperatively. e.g. How to construct, given any two numbers x and y, their sum? For any two sets x and y, how to construct their union? How to build the ultrapower of a model $M \models \mathsf{ZFC}$ an M-κ-complete ultrafilter U?.

- The next axiom concerns the determinacy axiom **Det** (just **D** in the paper).
- This was, recall: **Det.** $\langle i \rangle p \supset [i]p$ (there's only ever one way to execute an imperative.

- But lot's of imperatives aren't like this.
- e.g.1. "Given a finite line segment $l \subset \mathbb{R}$, extend l!"
- This isn't determinate (many l I could extend to).
- The natural ordering of such lines under the relation of being a subline is not-well-ordered (could go left, could go right), and in fact non-well-founded (could always have a shorter line)!
- e.g.2. "Given a model $M \models \mathsf{ZFC}$, and a forcing partial order $\mathbb{P} \in M$, add a $\mathbb{P}$-M-generic!".
- For many kinds of forcing (e.g. even the addition of a single Cohen real) we have all of indeterminacy, non-well-order, and non-well-foundedness (the latter two under subset).
- This may link to providing an imperatival formulation of more algebraic mathematics.
- e.g. 3. "Build a group, then permute it!"

Objective.

Examine imperatival foundations that drops **Det** and develop an account of indeterminate constructions within it (with a view to various indeterminate mathematical constructions).

- My (limited!) understanding of constructive/intuitionistic mathematics is that it's mostly couched in the declarative idiom, with the logic modified.
- But of course constructivism and intuitionism are closely linked to computational/constructive ideas (e.g. BHK interpretation).
- There's also the question of how this all relates to the theory of computation and recursion theory (e.g. ordinal time Turing machines).

Objective.

Interpret the usual constructive-style mathematics in the imperatival idiom. Possibilities here in virtue of (a) the recent technology on mirroring and intuitionistic logics (recall we have modal operators, (b) the known (but not by me) links between HOL and constructive/computational mathematics (?)

- How about the really big stuff?
- Note the imperative "Whilst the ordinals are accessible, do powerset!" effectively builds us V_{κ_0}, where κ_0 is the least inaccessible.
- But powerset always grows the domain by one level.
- The command, "Whilst the ordinals are accessible, do powerset! Then, do powerset! Then, whilst the ordinals are accessible do powerset!" will build you V_{κ_1}; the next inaccessible rank.
- There's a question of how far this can take us.
- Especially in virtue of links with fixed points and the indescribability hierarchy/reflection.

Objective

Examine how imperatives can be used to construct large cardinals "from below".

- **Summing up:**
- You can provide a foundation for mathematics on the basis of ideas of imperatives and executability.
- I'm tempted by the idea that they're just an "equivalent way of stating the same facts".
- But the approach also seems somewhat distinctive, and there seem to be lots of interesting open questions, and it provides a way of formally systematising imperatival talk in mathematics.