

ENGINEERING SET-THEORETIC CONCEPTS

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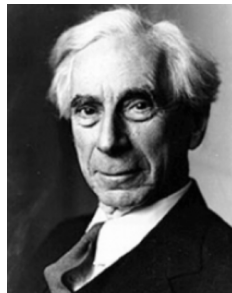
INTRODUCTION

- ▶ I thought I would start off with a **fairy tale**:
- ▶ Once upon a time, there was a logician named **Gottlob Frege**.
- ▶ He came up with a clever theory of **sets** viewed as **concept extensions**, and showed how you could do all sorts of **nice mathematical things**.



INTRODUCTION

- ▶ Unfortunately, his system was built **out of straw**.
- ▶ Along came the big bad **Bertrand Russell**.
- ▶ He huffed, and puffed, and blew the house down with **Russell's Paradox**.
- ▶ Later, we figured out the **right** fix to Russell's Paradox.
- ▶ We built a nice **brick** house out of the **iterative conception** of set and ZFC (our **favourite** theory of sets).
- ▶ We then **lived happily ever after** in our perfectly neat and tidy brick house made of iterative sets.



INTRODUCTION

OVERARCHING THEME:

This description is indeed a **fairy tale**. Whilst it might be a **comforting yarn** to tell the kids, it is **wrong**. Rather the reality of the matter is **far more intricate** (and **beautiful**) than this **simplistic** story.

MAIN CLAIMS:

- (1.) We are **now** at a **conceptual fork in the road**.
- (2.) In particular, whether **there are any uncountable sets at all** is **up for grabs**.

INTRODUCTION

NOT-SO-SECRET SECONDARY AIM:

Indicate some **connections** between these issues and **other** areas I'm working on.

- ▶ §1 Why set theory?
- ▶ §2 Iterative conceptions emerge
- ▶ §3 Maximality and the Cohen-Scott Paradox
- ▶ §4 Two roads forward
- ▶ §5 Conclusions and connections to other work

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§1 Why set theory?

- ▶ Set theory concerns a cluster of theories of **collections**, in particular those that are:
 - ▶ **Extensional.** Sets with the **same members** are **identical**.
 - ▶ **Objectual.** Sets are **objects** over and above their elements.
- ▶ At this point, we might wonder: **Why by interested in set theory at all?**
- ▶ Of course such collections are **perfectly good** as objects of philosophical study, but why has set theory occupied such a **central place** in our theorizing?
- ▶ One (**bad**) answer: Set theory provides our best **theory of collections**.
- ▶ But this **can't** be right, collection-talk **needn't** be **Objectual** **nor** need it be **Extensional**.

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- ▶ A better answer: *When given **good** axioms, set theory is able to **represent/encode** objects and problems, and provide systems with many **theoretical virtues**.*
- ▶ This has been studied deeply in the work of **Penelope Maddy** (see [Maddy, 1988a], [Maddy, 1988b], [Maddy, 2017], [Maddy, 2019]) and I've added some virtues in [Barton, Ele].



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- ▶ The ones that will be relevant for us **today** are:
- ▶ Set theory is a **Testing Ground for Paradox** in that it gives examples of many interesting **inconsistencies**, and allows us to **diagnose** them (we'll isolate this later as its own virtue that I'll call **Paradox Diagnosis**).
- ▶ Set theory provides a **Generous Arena** for mathematics—almost any mathematical object can be **represented/encoded** by sets.
- ▶ Set theory also provides **Risk Assessment**—if I can **encode** some other theory T using set theory, any underlying belief we have in **consistency of set theory** transfers immediately to T.
- ▶ Finally, set theory provides a **Theory of Infinity**—it is an important **family of theories** we use for studying **infinity** and its **arithmetic**.
- ▶ The 'standard' set theory we use is **Zermelo-Fraenkel set theory with the Axiom of Choice** or **ZFC**.
- ▶ ZFC performs **beautifully** with respect to these constraints (though as we'll see **not** perfectly).

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§2 Iterative conceptions emerge

- ▶ A **conception of set** is a story about **what sets are** that is designed to motivate a **good theory** for us, in line with these **theoretical virtues**. Compare (for example) Rawls on **conceptions of justice**. **Note:** Lots to say about concepts and conceptions, please **ask** in Q&A!
- ▶ We'll talk about the **principles** that a particular conception validates, these can be **formal** but also something more **informal**.
- ▶ e.g. Conceptions of **fairness** in terms of **outcome** vs. conceptions of fairness in terms of **effort** (see [Incurvati, 2017]).
- ▶ The notion of conception is **relative**. e.g. the **societal-benefit conception** of **fairness-by-outcome** vs the **revenue conception** of **fairness-by-outcome** (see [Barton, Eng]).

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- ▶ **Warm-up:** The naive conception of truth holds that the following principles hold:
- ▶ **Truth-theoretic ascent:** $\phi \rightarrow Tr(\phi)$
- ▶ **Truth-theoretic descent:** $Tr(\phi) \rightarrow \phi$.
- ▶ Ascent and descent, when combined with classical logic, yield a contradiction—the naive conception of truth is inconsistent!
- ▶ [Scharp, 2013] (controversially!) argues that the naive conception of truth should be replaced with two conceptions of truth: ascending truth and descending truth (with the corresponding principles holding of each).
- ▶ The important point for us: When faced with an inconsistent conception one option is to trade off principles against one another.

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- ▶ Let's consider the **first** contender for a conception of set:

THE NAIVE CONCEPTION OF SET

The **naive conception of set** holds that sets are **extensions of arbitrary predicates**.

- ▶ This motivates the **naive comprehension schema**, for **every condition** $\phi(x)$, **there is a set** of all the ϕ s.

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- ▶ Unfortunately using the condition $x \notin x$ we get a **contradiction** (this is **Russell's Paradox**).
- ▶ **Another** way of looking at Russell's Paradox (call it the **Cantor-Russell reasoning**).
- ▶ The condition $x = x$ yields the **universal set**.
- ▶ For any set u , $x \subseteq u$ is a **perfectly fine condition** and so we get the **powerset of x** (i.e. the set of all subsets) for any set x (we'll denote this by " $\mathcal{P}(x)$ ").
- ▶ **Cantor's Theorem** (closely linked to Russell's Paradox) tells us that $\mathcal{P}(x)$ is always **bigger** than x (in the sense that there's no bijection between x and $\mathcal{P}(x)$).
- ▶ So you can't have a **universal set** so long as you can **prove Cantor's Theorem** and **have powersets**—the powerset of the universal set would have to be simultaneously **bigger than** and **no bigger than** the universal set.

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- ▶ Recently [Incurvati, 2020] has looked at the following way of seeing the set-theoretic paradoxes:

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A conception C is **universal** iff there exists **a set of all the things falling under C** .

INDEFINITE EXTENSIBILITY

A conception C is **indefinitely extensible** iff whenever we succeed in defining a **set u of objects falling under C** , there is **an operation** which, given u , produces an **object falling under C but not belonging to u** .

- ▶ The naive conception (via the naive comprehension schema) licences **both**. Bad news.

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- ▶ Similar to **truth**, **Universality** and **Indefinite Extensibility** can be traded off—there are conceptions that validate **one** but **not** the other.
- ▶ **Combinatorial** conceptions hold that sets are formed out of **available pluralities**, **logical** conceptions hold that sets are formed using **well-defined predicates**.
- ▶ **Combinatorial** conceptions (tend to) **reject Universality** and **accept Indefinite Extensibility**, **logical** conceptions (tend to) **reject Indefinite Extensibility** and **accept Universality**.

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- ▶ Consider then our **intellectual ancestors**, faced with the paradoxes.
- ▶ Was one of **Universality** or **Indefinite Extensibility** somehow **latent** in their thought?
- ▶ Cantor remarks that a set is:
...many, which can be thought of as one, i.e., a totality of definite elements that can be combined into a whole by a law.
[Cantor, 1883, p. 916]

However he **also** says:

The totality of all [sets] cannot be conceived as a determinate, well-defined, and also a finished set (Cantor in 1897 correspondence to Hilbert)

- ▶ The **former** quotation meshes better with **Universality** and the **latter** with **Indefinite Extensibility**.

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- ▶ This is a **controversial** interpretation, but examples can be **multiplied** (e.g. Zermelo, Mirimanoff, see [Barton, Eng]). Potter sums it up **nicely**:

*...in an attempt to make the history of the subject read more like an **inevitable convergence on the one true religion**, some authors have tried to find evidence of the iterative conception **quite far back** in the history of the subject. [Potter, 2004, p. 36]*

- ▶ Rather than say that our intellectual ancestors were inevitably going to select one **conception**, I suggest that they found themselves at a **conceptual fork in the road**.

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In the end though, the following (combinatorial) conception of set emerged as the 'mainstream' one (in some sense):

THE ITERATIVE CONCEPTION OF SET

The **iterative conception of set** holds that sets are **formed in stages** out of some **initial starting objects**. At subsequent stages we form sets out of **previously available** sets using some **given operations**.

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But the iterative conception is **ambiguous** between:

THE WEAK ITERATIVE CONCEPTION OF SET

The **weak iterative conception of set** holds that sets are formed in stages from **some starting objects** using **some operations**, but we do **not** assume that **all possible subclasses of a given stage** are formed as sets at subsequent stages.

THE STRONG ITERATIVE CONCEPTION OF SET

The **strong iterative conception of set** holds that sets are formed in stages, starting with an initial starting set of objects. At subsequent stages we **form all possible subclasses of a given stage** as sets.

Note: The strong iterative conception is a **sharpening** of the weak iterative conception.

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- ▶ The **strong** iterative conception essentially holds that the universe is formed by iterating the **powerset operation** (and bundling everything we have together at limit stages).
- ▶ This can be formalised by axiomatising the notion of a stage **directly**, but there are also **modal** formulations (see, for example [Studd, 2013], [Button, 2021]).
- ▶ Starting with the **empty set**, we have a single operation **Reify!**, that takes **all the pluralities** in a stage and **reifies them into sets**.
- ▶ Using this modal theory, one can **motivate** ZFC (in the sense that the modal theory interprets ZFC), and thus a degree of **Risk Assessment**.
- ▶ We also get another theoretical virtue. **Paradox Diagnosis**: Since we get **new** sets at each additional stage using **Reify!**, **Indefinite Extensibility holds** and **Universality fails**.

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- ▶ The **strong** iterative conception has become something of the **assumed default** (perhaps a motivation for our initial **comforting story**).
- ▶ However, the history of set theory is **replete** with examples of the **weak** iterative conception.
- ▶ Some are quite **mathematically involved** so I won't go into them here (e.g. **relative constructibility**).
- ▶ The rough idea can be seen with the **hereditarily finite sets**, however.

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- ▶ One way of obtaining the hereditarily finite sets is just to iterate **powerset** through the natural numbers...
- ▶ But I could also iterate **Power-n!** which forms all subsets of **size at most n**.
- ▶ This is the same for the first few stages, but then **takes time to catch up**.
- ▶ We can think of **families of operations** for obtaining these sets that **aren't even well-ordered**.
- ▶ e.g. **Even!** forms all subsets of **even size**, **Odd!** forms all subsets of **odd size** (at a given stage).
- ▶ To get all the hereditarily finite sets, we have to **interleave Even!** and **Odd!** in a **sensible** way.

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§3 Maximality and the Cohen-Scott Paradox

- ▶ There is a **problem** though with the strong iterative conception and ZFC.
- ▶ It concerns **Theory of Infinity**.
- ▶ ZFC tells us **almost nothing** about the values of infinite sizes.

CONTINUUM HYPOTHESIS

The **continuum hypothesis** (or CH) says that there's **no** infinite set of reals **larger than** the natural numbers (0, 1, 2, 3...) but **smaller than** the real numbers (numbers you can represent with an infinite decimal, 0, 1, $\sqrt{2}$, π , ...).

- ▶ CH, along with **many** other statements can be **neither proved nor refuted** from ZFC (indeed so-called 'independence' is the **norm**).

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- ▶ **Forcing** is a way of **adding** sets to models of set theory, and is **central** in the proof that CH isn't provable from ZFC.
- ▶ **Important point 1:** You can make **any** set x countable using forcing.
- ▶ **Important point 2:** (Black-boxed) Forcing can be thought of as a **process** in it's own way, you can think of it as **throwing in** a new object into a model, and then closing under the operations definable there (in particular, the forcing extension is the **smallest** model containing both all elements of the ground model and your new set added).
- ▶ This is a bit like obtaining the **complex field** from the **field of real numbers**.
- ▶ As well as **Reify!**, we can think of forcing as providing another kind of command **Enumerate!** that adds **an enumeration** of a set via forcing.

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- ▶ **How to combat independence.** An idea that has been popular in set theory to combat independence is **Maximality**—the idea that there should be **as many different kinds of set** as possible.
- ▶ Unfortunately **Maximality** is too **vague** to do any significant work (see, among others [Barton, 2016], [Incurvati, 2017]).
- ▶ We'll look at the following idea:

THE FORCING-SATURATED CONCEPTION OF STRONGLY ITERATIVE SET

The forcing-saturated conception of strongly iterative set holds that the stages are formed by the **powerset operation** (via **Reify!**) and that there's **enough saturation under forcing** to support **Enumerate!** for any set x .

Note: There's a way of getting to this conception via **absoluteness** (the idea that if a kind of set **could exist** then one **does exist**). See [Barton, Eng], [Barton and Friedman, MS], or **ask** in Q&A!

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It will be useful to distinguish between:

POWERSSET

The **Powerset Axiom**—the axiom that any set x has a powerset—**holds**.

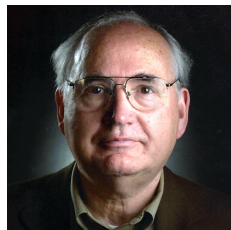
FORCING SATURATION

Any set can be **enumerated** using **forcing** (via **Enumerate!**).

The Cohen-Scott Paradox. **Powerset** pushes in the direction of **many very big (uncountable) cardinals** by Cantor's Theorem. But **Forcing Saturation** (via **Enumerate!**) implies that **every set is countable**. **Contradiction!** So the forcing-saturated conception of strongly iterative set is (without further modification) **inconsistent**.

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*I see that there are any number of contradictory set theories, all extending the Zermelo-Fraenkel axioms: but the models are all just models of the first-order axioms, and first-order logic is **weak**... Perhaps we would be pushed in the end to say that **all sets are countable** (and that the continuum is **not even a set**) when at last all cardinals are **absolutely destroyed**. [Scott, 1977, p. xv]*



§4 Two roads forward

- ▶ Is all **lost** for this kind of maximality in set theory? I say **No!**
- ▶ Recall **Truth-theoretic Ascent** and **Truth-theoretic Descent**.
- ▶ Recall **Universality** and **Indefinite Extensibility**.
- ▶ *Contrary* to our fairy tale (and perhaps ‘accepted wisdom’) just as our intellectual ancestors faced a **conceptual fork**, so do we **right now**.
- ▶ Should we accept **Forcing Saturation** or **Powerset**?
- ▶ There are two **competing** conceptions of **iterative set** here, the **forcing-saturated iterative conception** and **strongly iterative conception** of **set**.

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- ▶ How does the **forcing-saturated iterative conception** compare with the **strong iterative conception**?
- ▶ Since we have **Forcing Saturation**, every set is **countable** and we have to **drop** the Powerset Axiom (call this position **countabilism**).
- ▶ **Question.** What of the **iterative conception**?
- ▶ **Answer.** We can give **modal stage theories** for forcing-saturated conceptions, but they are **weakly** (and not **strongly**) iterative, and the stages are **not well-ordered**.
- ▶ Instead, like with **Even!** and **Odd!**, one can **Reify!** and **Enumerate!** **sensibly** to get the sets (see [Scambler, 2021]).
- ▶ In fact, if you start with **enough sets**, you **just** need **Enumerate!**.¹
- ▶ We still get **Paradox Diagnosis**, the operations and sets available at worlds **collaborate** to ensure that **Universality fails** and **Indefinite Extensibility holds**.

¹See [Barton, Ele], drawing on [Steel, 2014] and [Maddy and Meadows, 2020].

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- ▶ **Question.** What then of **ZFC**?
- ▶ **Answer.** There are nice axioms, drawing on ideas of **Forcing Saturation**, that imply that there are **inner models** of ZFC.²
- ▶ But in order to have ZFC, you have to **forget** about some **functions**.
- ▶ There is thus a kind of **symmetry** between the advocate of **Powerset** and the advocate of **Forcing Saturation**.
- ▶ People who believe **Forcing Saturation** think that advocates of **Powerset** miss out a bunch of **functions**.
- ▶ **Powerset** folks think that advocates of **Forcing Saturation** **miss out a bunch of large (uncountable) sets**.
- ▶ I've argued that there **may** be reasons (**surprisingly!**) to view the advocate of **Powerset** as doing something **more restrictive** (see [Barton, Res]).

²See [Barton and Friedman, MS].

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- ▶ With this situation in mind, let's revisit the forcing-saturated conception using our **theoretical virtues** from earlier (recalling that the advocate of **Powerset** can just **piggyback** off the standard picture, but has problems regarding **Theory of Infinity**).
- ▶ We do have **Risk Assessment** via our nice **modal stage theories** which provide an **intuitive background structure**.
- ▶ If we think that all proper classes are the same size, we get a **thorough Theory of Infinity**—every infinite class is either **countable** or **proper-class-sized** (i.e. the size of the **continuum**).
- ▶ **Generous Arena** is **more complicated**. The reals are a proper class, as is the class of all **continuous functions** from reals to reals, but the **usual representation** of all functions from reals to reals **doesn't exist**.
- ▶ However, you **can** get **simulacra** of large uncountable objects by **leaving out functions**.
- ▶ **Question**. Does the countabilist perform better with respect to other criteria (e.g. **Ontological Parsimony**)?

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§5 Conclusions and connections to other work

- ▶ So there we have it: **Contrary** to our comforting story and what one might assume, I think we're **now** at a conceptual fork in the road.
- ▶ In particular, down one fork there's **many uncountable sets** and down the other **every set is countable**.
- ▶ Whilst I acknowledge that the **strong** iterative conception is **further ahead** in the race, the **forcing-saturated** version of the **weak** iterative conception is **attractive**, and might **catch up**.
- ▶ Although this is quite a specific (albeit **fascinating**) problem in the philosophy of mathematics, I think there are **many** related problems, and I'd like to mention a few now.
- ▶ This will make the conclusion a bit **longer** than usual, I hope you'll indulge me in this.

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- ▶ There is a **host** of problems raised in the **philosophy of mathematics**.
 1. (*) Is one conception **better** or should we just **live with pluralism**?
 2. What about **other** conceptions of set? The enormous wealth of other theories **expanding ZFC**, **predicative** conceptions, **category-theoretic** or **schematic** conceptions...there is **a lot** of work to be done here.
 3. Some of the modal stage theories I've described have **ill-founded** accessibility relations. How to make the weak iterative conception **more precise**?
 4. One way to motivate both **Powerset** and **Forcing Saturation** is by considering **extensions** of the universe and **counterpossibles**.³ How should we think of these **philosophically**? How does this **connect** to other work on counterpossibles and impossible worlds?
 5. What about **absolute generality**? How should we think of operations like **Reify!** and **Enumerate!** (both over the **stages** and the **universe**).

³See [Barton, 2020], [Antos et al., 2021], and [Barton and Friedman, MS].

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- ▶ Connections to the **philosophy of science** and **pluralism**.
 1. (*) A **longer term** interest: Though I think (classical) mathematics has many **diverse** foundational viewpoints, they all exhibit **significant agreement** (e.g. regarding the natural numbers). It's perhaps worth assessing the prospects for a **perspectivism** in mathematics and **contrasting it** with what we find in the sciences (e.g. in the work of Stéphanie Ruphy, Michela Massimi, Ronald Giere, new collection [Massimi and McCoy, 2020]).
 2. I've pushed the idea that mathematics, though it may have its own methods, has its own questions of **choice** of **concepts/theory** as **other areas** of science. But how **strong** is this anti-exceptionalism?
 3. What about **conception pluralism** (as opposed to **perspectivism**) and how similar is it to the kinds of pluralism we see in the sciences (e.g. in the work of Nancy Cartwright and Hasok Chang)?

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- ▶ The conceptual subtlety we've seen relates to questions in **negative epistemology**:
 1. (*) **Fallibilism** and **epistemic luck** (esp. with respect to **Gettier cases** in mathematics, see [Barton, Get]).
 2. The nature of **ignorance** (we might not even understand **how** we are ignorant, see [Barton, 2017]).
 3. Issues regarding **suspension of judgement**. We might suspend because we are not sure a question **even has an answer** or at the **end of inquiry** (this can inform other accounts of suspension, e.g. Jane Friedman's view).

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- ▶ Relationship to work on **concepts** and **conceptual engineering**. (I'm looking at these questions in some in-progress work [Barton, Eng].)
 1. (*) Much of what I've said is indicative of **conceptual engineering**—the study of the (i) **evaluation**, (ii) **design**, and (iii) **implementation** of concepts. How does all this mesh with **other** engineering projects? e.g. Sally Haslanger's work regarding social justice and **amelioration** of concepts, the historical debate around Carnapian **explication**, Herman Cappelen's **linguistic** approach?
 2. (*) How much **control** do we have over our semantic whims? e.g. is significant **set-theoretic activism** either needed or desirable?
 3. What about different accounts of **concept individuation**? e.g. functional role, intensional equivalence, samesaying...
 4. What about **open texture**?

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Thanks for listening!

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