Engineering Set-Theoretic Concepts

Neil Barton

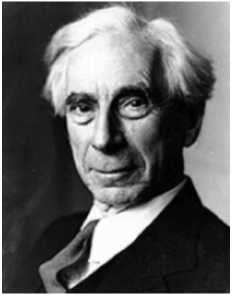
Slides available via the "Blog" section of my website
https://neilbarton.net/blog/

Introduction

- I thought I would start off with a fairy tale:
- Once upon a time, there was a logician named Gottlob Frege.
- He came up with a clever theory of sets viewed as concept extensions, and showed how you could do all sorts of nice mathematical things.

Introduction

- Unfortunately, his system was built out of straw.
- Along came the big bad Bertrand Russell.
- He huffed, and puffed, and blew the house down with Russell's Paradox.
- Later, we figured out the right fix to Russell's Paradox.
- We built a nice brick house out of the iterative conception of set and ZFC (our favourite theory of sets).
- We then lived happily ever after in our perfectly neat and tidy brick house made of iterative sets.



Introduction

- In this talk I want to convince you of the following:

Main Aims:

This description is indeed a fairy tale. Whilst it might be a comforting yarn to tell the kids, it doesn't represent how things actually went. Rather the reality of the matter is infinitely more beautiful than this simplistic story. In particular:

(1.) There are many twists and turns in the development of set theory.

(2.) Our intellectual ancestors faced different ways they might have gone.

(3.) We are now at a conceptual crossroads of our own.

(4.) In particular, whether there are any uncountable sets at all is up for grabs.

Introduction

Not-so-secret secondary aim:

Indicate some connections between these issues and other areas I'm working on.

- §1 Why set theory?
- §2 Iterative conceptions emerge
- §3 Maximality and the Cohen-Scott Paradox
- §4 Two roads forward
- §5 Conclusions and connections to other work

§1 Why set theory?

- Set theory concerns a cluster of theories of collections, in particular those that are:
  - **Extensional.** Sets with the same members are identical.
  - **Objectual.** Sets are objects over and above their elements.
- At this point, we might wonder: Why by interested in set theory at all?
- Of course such collections are perfectly good as objects of philosophical study, but why has set theory occupied such a central place in our theorizing?
- One (bad) answer: Set theory provides our best theory of collections.
- But this can't be right, collection-talk needn't be **Objectual** nor need it be **Extensional**.

Engineering set-theoretic concepts

- A better answer: *When given good axioms, set theory is able to represent/encode objects and problems, and provide systems with many theoretical virtues*.
- This has been studied deeply in the work of Penelope Maddy (see [Maddy, 1988a], [Maddy, 1988b], [Maddy, 2017], [Maddy, 2019]) and I've added some virtues in [Barton, Ele].

Engineering set-theoretic concepts

- The ones that will be relevant for us today are:
- Set theory is a **Testing Ground for Paradox** in that it gives examples of many interesting inconsistencies, and allows us to diagnose them.
- Set theory provides a **Generous Arena** for mathematics—almost any mathematical object can be represented/encoded by sets.
- Set theory also provides **Risk Assessment**—if I can encode some other theory T using set theory, any underlying belief we have in consistency of set theory transfers immediately to T.
- Finally, set theory provides a **Theory of Infinity**—it is an important family of theories we use for studying infinity and its arithmetic.
- The 'standard' set theory we use is Zermelo-Fraenkel set theory with the Axiom of Choice or ZFC.
- ZFC performs beautifully with respect to these constraints (though as we'll see not perfectly).

§2 Iterative conceptions emerge

- A conception of set is a story about what sets are that is designed to motivate a good theory for us, in line with these theoretical virtues. Compare (for example) Rawls on conceptions of justice. **Note:** Lots to say about concepts and conceptions, please ask in Q&A!
- We'll talk about the principles that a particular conception validates, these can be formal but also something more informal.
- e.g. Conceptions of fairness in terms of outcome vs. conceptions of fairness in terms of effort (see [Incurvati, 2017]).
- The notion of conception is relative. e.g. the societal-benefit conception of fairness-by-outcome vs the revenue conception of fairness-by-outcome (see [Barton, Eng]).

Engineering set-theoretic concepts

- **Warm-up:** The naive conception of truth holds that the following principles hold:
- **Truth-theoretic ascent:** $\phi \to Tr(\phi)$
- **Truth-theoretic descent:** $Tr(\phi) \to \phi$.
- Ascent and descent, when combined with classical logic, yield a contradiction—the naive conception of truth is inconsistent!
- [Scharp, 2013] (controversially!) argues that the naive conception of truth should be replaced with two conceptions of truth: ascending truth and descending truth (with the corresponding principles holding of each).
- The important point for us: When faced with an inconsistent conception one option is to trade off principles against one another.

Engineering set-theoretic concepts

- Let's consider the first contender for a conception of set:

The naive conception of set

The naive conception of set holds that sets are extensions of arbitrary predicates.

- This motivates the naive comprehension schema, for every condition with one free variable $\phi(x)$, there is a set of all the ϕs.

Engineering set-theoretic concepts

- Unfortunately using the condition $x \notin x$ we get a contradiction (this is Russell's Paradox).
- Another way of looking at Russell's Paradx (call it the Cantor-Russell reasoning).
- The condition $x = x$ yields the universal set.
- For any set u, $x \subseteq u$ is a perfectly fine condition and so we get the powerset of x (i.e. the set of all subsets) for any set x (we'll denote this by "$\mathcal{P}(x)$").
- **Cantor's Theorem** (closely linked to Russell's Paradox) tells us that $\mathcal{P}(x)$ is always bigger than x (in the sense that there's no bijection between x and $\mathcal{P}(x)$).
- So you can't have a universal set so long as you can prove Cantor's Theorem and have powersets—the powerset of the universal set would have to be simultaneously bigger than and no bigger than the universal set.

Engineering set-theoretic concepts

- Recently [Incurvati, 2020] has looked at the following way of seeing the set-theoretic paradoxes:

Universality

A conception C is universal iff there exists a set of all the things falling under C.

Indefinite Extensibility

A conception C is indefinitely extensible iff whenever we succeed in defining a set u of objects falling under C, there is an operation which, given u, produces an object falling under C but not belonging to u.

- The naive conception (via the naive comprehension schema) licences both. Bad news.

Engineering set-theoretic concepts

- Similar to truth, **Universality** and **Indefinite Extensibility** can be traded off—there are conceptions that validate one but not the other.
- **Combinatorial** conceptions hold that sets are formed out of available pluralities, **logical** conceptions hold that sets are formed using well-defined predicates.
- Combinatorial conceptions (tend to) reject **Universality** and accept **Indefinite Extensibility**, logical conceptions (tend to) reject **Indefinite Extensibility** and accept **Universality**.

Engineering set-theoretic concepts

- Consider then our intellectual ancestors, faced with the paradoxes.
- Was one of **Universality** or **Indefinite Extensibility** somehow latent in their thought?
- Cantor remarks that a set is:

  ...many, which can be thought of as one, i.e., a totality of definite elements that can be combined into a whole by a law. [Cantor, 1883, p. 916]

However he also says:

> *The totality of all [sets] cannot be conceived as a determinate, well-defined, and also a finished set (Cantor in 1897 correspondence to Hilbert)*

- The former quotation meshes better with **Universality** and the latter with **Indefinite Extensibility**.

Engineering set-theoretic concepts

- This is a controversial interpretation, but examples can be multiplied (e.g. Zermelo, Mirimanoff, see [Barton, Eng]). Potter sums it up nicely:

  ...in an attempt to make the history of the subject read more like an inevitable convergence on the one true religion, some authors have tried to find evidence of the iterative conception quite far back in the history of the subject. [Potter, 2004, p. 36]

- Rather than say that our intellectual ancestors were inevitably going to select one conception, I suggest that they found themselves at a conceptual crossroads.

- In the end though, the following (combinatorial) conception of set emerged as the 'mainstream' one (in some sense):

The iterative conception of set

The iterative conception of set holds that sets are formed in stages out of some initial starting objects. At subsequent stages we form sets out of previously available sets using some given operations.

- But the iterative conception is ambiguous between:

The weak iterative conception of set

The weak iterative conception of set holds that sets are formed in stages from some starting objects using some operations, but we do not assume that all possible sets are formed at subsequent stages.

The strong iterative conception of set

The strong iterative conception of set holds that sets are formed in stages, starting with an initial starting set of objects. At subsequent stages we form all possible subsets of every available set.

- **Note:** The strong iterative conception is a sharpening of the weak iterative conception.

Engineering set-theoretic concepts

- The strong iterative conception essentially holds that the universe is formed by iterating the powerset operation (and bundling everything we have together at limit stages).
- This can be formalised by axiomatising the notion of a stage directly, but there are also modal formulations (see, for example [Studd, 2013], [Button, 2021]).
- Starting with the empty set, we have a single operation **Reify!**, that takes all the pluralities in a stage and reifies them into sets.
- Using this modal theory, one can motivate ZFC (in the sense that the modal theory interprets ZFC), and thus a degree of **Risk Assessment**.
- We also get another theoretical virtue. **Paradox Diagnosis**: Since we get new sets at each additional stage using **Reify!**, **Indefinite Extensibility** holds and **Universality** fails.

Engineering set-theoretic concepts

- The strong iterative conception has become something of the assumed default (perhaps a motivation for our initial comforting story).
- However, the history of set theory is replete with examples of the weak iterative conception.
- Some are quite mathematically involved so I won't go into them here (e.g. relative constructibility).
- The rough idea can be seen with the hereditarily finite sets, however.

Engineering set-theoretic concepts

- One way of obtaining the hereditarily finite sets is just to iterate powerset through the natural numbers...
- But I could also iterate **Power-n!** which forms all subsets of size at most n.
- This is the same for the first few stages, but then takes time to catch up.
- We can think of families of operations for obtaining these sets that aren't even well-ordered.
- e.g. **Even!** forms all subsets of even size, **Odd!** forms all subsets of odd size (at a given stage).
- To get all the hereditarily finite sets, we have to interleave **Even!** and **Odd!** in a sensible way.

§3 Maximality and the Cohen-Scott Paradox

- There is a problem though with the strong iterative conception and ZFC.
- It concerns **Theory of Infinity**.
- ZFC tells us almost nothing about the values of infinite sizes.

Continuum Hypothesis

The continuum hypothesis (or CH) says that there's no infinite set of reals larger than the natural numbers (0, 1, 2, 3...) but smaller than the real numbers (numbers you can represent with an infinite decimal, 0, 1, $\sqrt{2}$, π,...).

- CH, along with many other statements can be neither proved nor refuted from ZFC (indeed so-called 'independence' is the norm).

Engineering set-theoretic concepts

- I'll give a brief flavour of some of the tricky mathematical ideas behind the proof.
- CH says that there are lots of kinds of **function** compared with the kinds of **sets of reals**—every infinite set of reals has a function that either bijects it with the natural numbers or with the reals.
- $\neg$CH by contrast, says that there are lots of kinds of **sets of reals** as compared with kinds of **function**—there's some infinite set of reals x for which there's no bijection between x and the naturals nor a bijection between x and the reals.
- Forcing lets you add sets to models, whilst preserving ZFC.
- So for $\neg$CH, we add a bunch of reals to a model of ZFC.
- Suppose then that $\neg$CH holds. What could you add to restore CH?...Add a bunch of functions! In particular make some sets countable again.

Engineering set-theoretic concepts

- **Important point 1:** You can make any set x countable using forcing.
- **Important point 2:** (Black-boxed) Forcing can be thought of as a process in it's own way, you can think of it as throwing in a new object into a model, and then closing under the operations definable there (in particular, the forcing extension is the smallest model containing both all elements of the ground model and your new set added).
- This is a bit like obtaining the complex field from the field of real numbers.
- As well as **Reify!**, we can think of forcing as providing another kind of command **Enumerate!** that adds an enumeration of a set via forcing.

Engineering set-theoretic concepts

- **How to combat independence.** An idea that has been popular in set theory to combat independence is **Maximality**—the idea that there should be as many different kinds of set as possible.
- Unfortunately **Maximality** is too vague to do any significant work (see, among others [Barton, 2016], [Incurvati, 2017]).
- We'll look at the following idea:

The forcing-saturated conception of strongly iterative set

The forcing-saturated conception of strongly iterative set holds that the stages are formed by the powerset operation (via **Reify!**) and that there's enough saturation under forcing to support **Enumerate!** for any set x.

Note: There's a way of getting to this conception via absoluteness (the idea that if a kind of set could exist then one does exist). See [Barton, Eng], [Barton and Friedman, MS], or ask in Q&A!

Engineering set-theoretic concepts

It will be useful to distinguish between:

Powerset

The Powerset Axiom—the axiom that any set x has a powerset—holds.

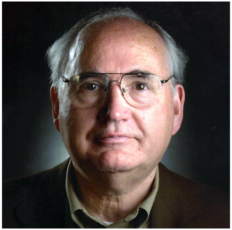
Forcing Saturation

Any set can be enumerated using forcing (via **Enumerate!**).

The Cohen-Scott Paradox. **Powerset** pushes in the direction of many very big (uncountable) cardinals by Cantor's Theorem. But **Forcing Saturation** (via **Enumerate!**) implies that every set is countable. Contradiction! So the forcing-saturated conception of strongly iterative set is (without further modification) inconsistent.

Engineering set-theoretic concepts

I see that there are any number of contradictory set theories, all extending the Zermelo-Fraenkel axioms: but the models are all just models of the first-order axioms, and first-order logic is weak... Perhaps we would be pushed in the end to say that all sets are countable (and that the continuum is not even a set) when at last all cardinals are absolutely destroyed. [Scott, 1977, p. xv]



§4 Two roads forward

- Is all lost for this kind of maximality in set theory? I say No!
- Recall **Truth-theoretic Ascent** and **Truth-theoretic Descent**.
- Recall **Universality** and **Indefinite Extensibility**.
- **Contrary** to our fairy tale (and perhaps 'accepted wisdom') just as our intellectual ancestors faced a conceptual crossroads, so do we right now.
- Should we accept **Forcing Saturation** or **Powerset**?
- There are two competing conceptions of iterative set here, the forcing-saturated iterative conception and strongly iterative conception of set.

- Is one significantly better than the other?
- Recall our theoretical virtues from earlier...
- The strongly iterative conception can essentially piggyback off the very nice 'default' story we discussed for ZFC.
- But many problems regarding **Theory of Infinity** remain.
- Things are not so simple for the advocate of forcing saturation.

Engineering set-theoretic concepts

- Since we have **Forcing Saturation**, every set is countable and we have to drop the Powerset Axiom (call this position countabilism).
- **Question.** What of the iterative conception?
- **Answer.** We can give modal stage theories for forcing-saturated conceptions, but they are weakly (and not strongly) iterative, and the stages are not well-ordered.
- Instead, like with **Even!** and **Odd!**, one can **Reify!** and **Enumerate!** sensibly to get the sets (see [Scambler, 2021]).
- In fact, if you start with enough sets, you just need **Enumerate!**.[1]
- We still get **Paradox Diagnosis**, the operations and sets available at worlds collaborate to ensure that **Universality** fails and **Indefinite Extensibility** holds.

[1] See [Barton, Ele], drawing on [Steel, 2014] and [Maddy and Meadows, 2020].

Engineering set-theoretic concepts

- **Question.** What then of ZFC?
- **Answer.** There are nice axioms, drawing on ideas of **Forcing Saturation**, that imply there are inner models of ZFC.[2]
- But in order to have ZFC, you have to forget about some functions.
- There is thus a kind of symmetry between the advocate of **Powerset** and the advocate of **Forcing Saturation**. **Forcing Saturation** folks think that advocates of **Powerset** miss out a bunch of functions. **Powerset** folks think that advocates of **Forcing Saturation** miss out a bunch of large (uncountable) sets.
- I've argued that there may be reasons (surprisingly!) to view the advocate of **Powerset** as doing something more restrictive (see [Barton, Res]).

[2] See [Barton and Friedman, MS].

Engineering set-theoretic concepts

- With this situation in mind, let's revisit the forcing-saturated conception using our theoretical virtues from earlier (recalling that the advocate of **Powerset** can just piggyback off the standard picture, but has problems regarding **Theory of Infinity**).
- We do have **Risk Assessment** via our nice modal stage theories which provide an intuitive background structure.
- We provide a thorough **Theory of Infinity**—every infinite class is either countable or proper-class-sized (i.e. the size of the continuum).
- Does this mesh better with the kinds of infinity we seem to encounter in the natural sciences?
- **Generous Arena** is more complicated. The reals are a proper class, as is the class of all continuous functions from reals to reals, but the usual representation of all functions from reals to reals doesn't exist.
- However, you can get simulacra of large uncountable objects by leaving out functions.

§5 Conclusions and connections to other work

- So there we have it: Contrary to our comforting story and what one might assume, I think we're now at a conceptual crossroads.
- In particular, down one fork there's many uncountable sets and down the other every set is countable.
- Whilst I acknowledge that the strong iterative conception is further ahead in the race, the forcing-saturated version of the weak iterative conception is attractive, and might catch up.
- Although this is quite a specific (albeit fascinating) problem in the philosophy of mathematics, I think there are many related problems, and I'd like to mention a few now.
- This will make the conclusion a bit longer than usual, I hope you'll indulge me in this.

Engineering set-theoretic concepts

- There is a host of problems raised in the philosophy of mathematics.
  (1.) (∗) Is one conception better or should we just live with pluralism? What does this mean for mathematical truth?
  (2.) What about other conceptions of set? The enormous wealth of other theories expanding ZFC, predicative conceptions, category-theoretic or schematic conceptions...there is a lot of work to be done here.
  (3.) Some of the modal stage theories I've described have ill-founded accessibility relations. How to make the weak iterative conception more precise?
  (4.) One way to motivate both **Powerset** and **Forcing Saturation** is by considering extensions of the universe and counterpossibles.[3] How should we think of these philosophically? How does this connect to other work on counterpossibles and impossible worlds?
  (5.) What about absolute generality? How should we think of operations like **Reify!** and **Enumerate!** (both over the stages and the universe).

[3] See [Barton, 2020], [Antos et al., 2021], and [Barton and Friedman, MS].

Engineering set-theoretic concepts

- Connections to the philosophy of science and pluralism.
  1. (∗) A longer term interest: Though I think (classical) mathematics has many diverse foundational viewpoints, they all exhibit significant agreement (e.g. regarding the natural numbers). It's perhaps worth assessing the prospects for a perspectivism in mathematics and contrasting it with what we find in the sciences (e.g. in the work of Stéphanie Ruphy, Michela Massimi, Ronald Giere, new collection [Massimi and McCoy, 2020]).
  2. I've pushed the idea that mathematics, though it may have its own methods, has its own questions of choice of concepts/theory as other areas of science. But how strong is this anti-exceptionalism?
  3. What about conception pluralism (as opposed to perspectivism) and how similar is it to the kinds of pluralism we see in the sciences (e.g. in the work of Nancy Cartwright and Hasok Chang)?

Engineering set-theoretic concepts

- The conceptual subtlety we've seen relates to questions in negative epistemology:
  1. ($*$) The nature of ignorance (we might not even understand how we are ignorant, see [Barton, 2017]).
  2. Fallibilism and epistemic luck (esp. with respect to Gettier cases in mathematics, see [Barton, Get]).
  3. Issues regarding suspension of judgement. We might suspend because we are not sure a question even has an answer or at the end of inquiry (this can inform other accounts of suspension, e.g. Jane Friedman's view).

Engineering set-theoretic concepts

- Relationship to work on concepts and conceptual engineering. (I'm looking at these questions in some in-progress work [Barton, Eng].)
  1. (∗) Much of what I've said is indicative of conceptual engineering—the study of the (i) evaluation, (ii) design, and (iii) implementation of concepts. How does all this mesh with other engineering projects? e.g. Sally Haslanger's work regarding social justice and amelioration of concepts, the historical debate around Carnapian explication, Herman Cappelen's linguistic approach?
  2. (∗) How much control do we have over our semantic whims? e.g. is significant set-theoretic activism either needed or desirable?
  3. What about different accounts of concept individuation? e.g. functional role, intensional equivalence, samesaying...
  4. What about open texture?

Thanks for listening!

References I

Antos, C., Barton, N., and Friedman, S.-D. (2021).
Universism and extensions of V.
The Review of Symbolic Logic, 14(1):112–154.

Barton, N. (2016).
Richness and reflection.
Philosophia Mathematica, 24(3):330–359.

Barton, N. (2017).
Independence and ignorance: How agnotology informs set-theoretic pluralism.
Journal of Indian Council of Philosophical Research, 34(2):399–413.

Barton, N. (2020).
Forcing and the universe of sets: Must we lose insight?
Journal of Philosophical Logic, 49(4):575–612.

Barton, N. (Ele).
Iterative Conceptions of Set.
Cambridge University Press.
Manuscript under review. Preprint:
https://neilbartonphilosophy.files.wordpress.com/2023/01/iterative_conceptions_of_set.pdf.

Barton, N. (Eng).
Engineering Set-Theoretic Concepts.
Manuscript in preparation. Draft:
https://neilbartonphilosophy.files.wordpress.com/2022/12/engineering_set_theoretic_concepts.pdf.

References II

Barton, N. (Get).
Mathematical gettier cases and their implications.
Manuscript under review. Preprint: https://philpapers.org/rec/BARMGC-4R.

Barton, N. (Res).
Is (un)countabilism restrictive?
Manuscript under review. Preprint: https://philpapers.org/rec/BARIUR.

Barton, N. and Friedman, S. (MS).
Countabilism and maximality principles.
Manuscript under review. Preprint: https://philpapers.org/rec/BARCAM-5.

Bell, J. (2011).
Set Theory: Boolean-Valued Models and Independence Proofs.
Oxford University Press.

Button, T. (2021).
Level theory, part 2: Axiomatizing the bare idea of a potential hierarchy.
Bulletin of Symbolic Logic, 27(4):461–484.

Caicedo, A. E., Cummings, J., Koellner, P., and Larson, P. B., editors (2017).
Foundations of Mathematics: Logic at Harvard Essays in Honor of W. Hugh Woodin's 60th Birthday, volume 690 of Contemporary Mathematics.
American Mathematical Society.

References III

Cantor, G. (1883).
Foundations of a general theory of manifolds; a mathematico-philosophical investigation into the theory of the infinite.
In Ewald, W., editor, From Kant to Hilbert; A Source Book in the Foundations of Mathematics, volume 2. Oxford University Press.

Incurvati, L. (2017).
Maximality principles in set theory.
Philosophia Mathematica, 25(2):159–193.

Incurvati, L. (2020).
Conceptions of Set and the Foundations of Mathematics.
Cambridge University Press.

Maddy, P. (1988a).
Believing the axioms. I.
The Journal of Symbolic Logic, 53(2):481–511.

Maddy, P. (1988b).
Believing the axioms II.
The Journal of Symbolic Logic, 53(3):736–764.

Maddy, P. (2017).
Set-theoretic foundations.
In [Caicedo et al., 2017], pages 289–322. American Mathematical Society.

References IV

Maddy, P. (2019).
What Do We Want a Foundation to Do?, pages 293–311.
Springer International Publishing, Cham.

Maddy, P. and Meadows, T. (2020).
A reconstruction of Steel's multiverse project.
The Bulletin of Symbolic Logic, 26(2):118–169.

Massimi, M. and McCoy, C., editors (2020).
Understanding Perspectivism: Scientific Challenges and Methodological Prospects.
Routledge.

Potter, M. (2004).
Set Theory and its Philosophy: A Critical Introduction.
Oxford University Press.

Scambler, C. (2021).
Can all things be counted?
Journal of Philosophical Logic.

Scharp, K. (2013).
Replacing Truth.
Oxford University Press.

Scott, D. (1977).
Foreword to *boolean-valued models and independence proofs*.
In [Bell, 2011], pages xiii–xviii. Oxford University Press.

References V

Steel, J. (2014).
Gödel's program.
In Kennedy, J., editor, Interpreting Gödel. Cambridge University Press.

Studd, J. P. (2013).
The iterative conception of set: A (bi-) modal axiomatisation.
Journal of Philosophical Logic, 42(5):697–725.