Algebraic Levels in Mathematical Structuralism

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Introduction

- This talk concerns how model theory can inform our thinking concerning the notion of structure in philosophy.
- But first, I want to start with some thanks.
  1. Thank you for the opportunity to talk!
  2. Thanks to Moritz Müller for teaching a fab model theory course back while I was at the KGRC.
  3. Thanks to John Baldwin and Andrés Villaveces for many patient discussions about model theory.

Introduction

- This talk concerns how we talk about mathematical structure.
- Quite often, contemporary model theory is regarded as orthogonal to the study of mathematical structure (particularly in the context of structuralism—the thesis that the subject matter of mathematics is structure):

  > *The point is that the model-theoretic notion of structure takes as its starting point a domain of objects and is a construction (definition) within set theory with urelemente, or within pure set theory. insofar as the notion of mathematical object is philosophically problematic, appeal to this account begs the question. ([Isaacson, 2011], p. 26)*

But we should distinguish between model theory as providing an ontological foundation for structuralism and being informative for understanding how we talk about structure.

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- However, model theory does help us see the following distinction:

Non-algebraic theory

A theory T is *non-algebraic* (or *categorical*) iff it has exactly one model (up to isomorphism).

Algebraic theory

A theory T is *algebraic* iff it many non-isomorphic models.

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Main Claim.

Model theory is important for understanding better how we talk about structure. In particular, it helps us see that there are kinds of theory that are algebraic, but not in the same sense as the group axioms or first-order arithmetic.

- §1 LEGO^{TM}-like theories
- §2 Strong minimality
- §3 Squeezing LEGO?
- §4 The LEGO-hierarchy/Conclusions/Questions

§1 LEGO$^{\text{TM}}$-like theories

- Consider the following theory (that I could write in first-order logic): "I consist solely of independent two-cycles."
- This isn't a non-algebraic theory talking about a determinate structure.
- But it does have a particular structure as its base, with an instruction for how a model should be built up (just repeat!).

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Definition.

(Informal) A theory is *LEGO-like* iff it encodes:

- Our LEGO-blocks: Some base structure(s), and
- An instruction manual: Given some cardinal κ, a precise set of instructions to build up a model of T of size κ from this initial template.

Philosophical Aside: Are there LEGO-like structures too?

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- Some (boring) examples of LEGO-like theories:
- Any categorical theory.
- Some quasi-categorical theories. (e.g. ZFC_2).
- Some silly almost categorical theories (e.g. ZFC_2 + "There are no inaccessible cardinals" $\vee$ PA_2).

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- Note all those theories are somehow higher-order.
- Normally first-order logic is taken to be unable to tell us anything interesting about infinite structure.
- This a quick consequence of the nice meta-theoretic properties (e.g. Compactness, Löwenheim-Skolem)—they can never be non-algebraic/categorical for infinite structures.
- But can they be LEGO-like?

§2 Strong minimality

I contend that strongly minimal theories are LEGO-like. We'll need some definitions for this.

Definition.

Let $\mathbb{G}$ be a set and $cl : \mathcal{P}(\mathbb{G}) \rightarrow \mathcal{P}(\mathbb{G})$ be a function (the closure operation). Then $(\mathbb{G}, cl)$ is a *pre-geometry* iff:

(I) $A \subseteq cl(A)$ and $cl(cl(A)) = cl(A)$.

(II) If $A \subseteq B$ then $cl(A) \subseteq cl(B)$.

(III) If $a \in cl(A \cup \{b\}) \backslash cl(A)$ then $b \in cl(A \cup \{A\})$.

(IV) If $a \in cl(A)$ then there is a finite $A_0 \subseteq A$ such that $a \in cl(A_0)$.

This is a generalisation of the notion of algebraic closure from the theory of fields.

Definitions.

If $(\mathbb{G}, cl)$ is a pre-geometry then:

(i) A set $B \subseteq \mathbb{G}$ is independent iff $c \notin cl(B\backslash\{c\})$ for all $c \in B$.

(ii) A set $A \subseteq \mathbb{G}$ is closed iff $A = cl(A)$.

(iii) A subset B of a closed set A is a basis of A iff B is independent and $cl(B) = A$.

(iv) The dimension of a closed set A is the cardinality of any basis of A.

These definitions effectively give generalisations of what you get with garden-variety spaces like Euclidean space: The dimension tells you how many coordinates are needed to specify a point in the geometry.

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Definitions.

- Let T be countable, complete, first-order, infinitely satisfiable (these will be suppressed from now on) and consider $\mathfrak{M} \models \mathrm{T}$, $M = dom(\mathfrak{M})$. Let

$$\phi(\mathfrak{M}) = \{\bar{a} \in M^n | \mathfrak{M} \models \phi(\bar{a})\}$$

  be any infinite definable subset in $\mathfrak{M}$. Then $\phi(\mathfrak{M})$ is *minimal in $\mathfrak{M}$* iff for all $\mathscr{L}(\mathfrak{M})$-formulas $\psi(\bar{x})$ the intersection $\phi(\mathfrak{M}) \cap \psi(\mathfrak{M})$ is either finite or cofinite in $\phi(\mathfrak{M})$.
- A formula $\phi(\bar{x})$ is *strongly minimal* iff $\phi(\bar{x})$ defines a minimal set in every elementary extension $\mathfrak{N}$ of $\mathfrak{M}$ (and we also say that $\phi(\mathfrak{M})$ is *strongly minimal* in this case).
- A theory T is *strongly minimal* if the formula $x = x$ is strongly minimal.

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- Given a strongly minimal set $\mathbb{G} = \phi(\mathfrak{M})$, it will be defined using parameters from some finite A_0.
- We can then define a closure operation $cl(A) = acl(A \cup A_0) \cap \mathbb{G}$, where $acl(B)$ is the model-theoretic notion of algebraic closure, i.e. the set of elements $c \in M$ s.t. there is a formula $\psi(x)$ with parameters from B such that $\mathfrak{M} \models \psi(c)$ and only finitely many elements of M satisfy $\psi(x)$ in $\mathfrak{M}$.

Fact.

Given these definitions $(\mathbb{G}, cl)$ is a pre-geometry.

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- We can think of a strongly minimal theory as LEGO-like: The strongly minimal set given by the formula $x = x$ provides our base structure and the pre-geometry is our instruction manual for generating new structures given some cardinal base κ.
- But it's not determinate which base cardinality κ we pick.

§3 Squeezing LEGO?

- We now have the following question: Are the LEGO-like structures exactly the strongly minimal ones?
- The idea of squeezing:

Sufficient formal class $\subseteq$ Informal class $\subseteq$ Necessary formal class

Provable $\subseteq$ Informally valid $\subseteq$ Valid on every structure

Turing computable $\subseteq$ Effectively computable $\subseteq$ KU-Computable

Strongly minimal $\subseteq$ LEGO-like $\subseteq$???

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Definitions.

- T is *categorical in κ* iff T has exactly one (up to iso) model of size κ.
- T is *totally categorical* iff it is categorical in every infinite cardinal.

- Perhaps totally categorical could serve as our necessity class (i.e. LEGO-like $\subseteq$ totally categorical)?

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- Signs are initially encouraging, since if T is totally categorical then it's categorical in $\aleph_1$, and...

Theorem.

Morley's Theorem. If T is categorical in $\aleph_1$, then it is categorical in every uncountable cardinal.

Moreover...

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Theorem.

(Implicit in Baldwin-Lachlan proof of Morley's Theorem) Suppose that **T** is uncountably categorical. Then **T** has a countable model $\mathfrak{M}$ with a strongly minimal set $\mathbb{G}$ such that:

(1.) For any model $\mathfrak{N} \models \mathbf{T}$ there is an elementary embedding $j : \mathfrak{M} \to \mathfrak{N}$.

(2.) Any model $\mathfrak{N} \models \mathbf{T}$ of uncountable cardinality λ has $dim(\mathbb{G}(\mathfrak{N})) = \lambda$.

(3.) Any models $\mathfrak{N}, \mathfrak{N}'$ of **T** with $dim(\mathbb{G}(\mathfrak{N})) = dim(\mathbb{G}(\mathfrak{N}'))$ are isomorphic.

- So there's a sense in which uncountably categorical theories admit of a kind of LEGO-like-ness too.
- This is desperately close to a squeezing argument...sadly...

- **Problem 1.** (Andrés Villaveces) There are strongly minimal theories that are not totally categorical.
- e.g. ACF_0 (theory of algebraically closed fields of characteristic 0) is strongly minimal but only uncountably categorical.
- But. What is missing?
- Exactly the fact that there are countably many non-isomorphic countable models, all of them classified by dimension (part of the content of Baldwin-Lachlan).

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- **Problem 2.** There are uncountably categorical theories that are not strongly minimal.
- An example due to Noah Schweber: Consider the theory of two equal-cardinality infinite sets:
- Language has two unary predicates U and V and a binary relation E.
- The theory says that U and V partition the universe, that E defines a bijection between U and V, and that the universe is infinite.
- This is in fact totally categorical, but not strongly minimal — indeed, every model has infinite coinfinite definable sets.
- This theory is almost strongly minimal though (Intuitively: The elements of an arbitrary M can be coordinatized by elements of a strongly minimal subset of that model.)

§4 The LEGO-hierarchy/Conclusions/Questions

Where to go from here?

- I had hoped to go through a huge hierarchy of different theories.
- We have almost strongly minimal, stable, ω-stable,...
- Sadly, tempus fugit.
- Maybe LEGO-like-ness is more of a family resemblance idea?
- I do think the quasi-squeeze shows that there's an important class of theories, hovering around almost strongly minimal and uncountably categorical.
- More generally, there's a large number of distinctions to be made here (e.g. [Morales et al., 2019] argue that the stability hierarchy measures *distance* from uniqueness in some sense).
- Can we isolate a sensible class here?

If there's time, a fascinating phenomenon:

Conjecture.

The Zilber Trichotomy Conjecture (roughly speaking), states that the geometry of every strongly minimal set is either (i) trivial, (ii) vector-space-like (modular), or (iii) field-like (non-modular).

As it turns out the conjecture is false [Hrushovski, 1993] gave an example of a strongly minimal set that did not fit this template.

Thanks

Thanks for listening, I look forward to the comments!

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Hrushovski, E. (1993).
A new strongly minimal set.
Annals of Pure and Applied Logic, 62(2):147 – 166.

Isaacson, D. (2011).
The reality of mathematics and the case of set theory.
In Noviak, Z. and Simonyi, A., editors, Truth, Reference, and Realism, pages 1–75. Central European University Press.

Morales, J. A. C., Villaveces, A., and Zilber, B. (2019).
Around logical perfection.