Engineering Set-Theoretic Concepts

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Introduction

- This talk is about a mini-book I've been writing on conceptual engineering and set theory.
- One thing I want to do with the book is provide an intuitive account of some of the very technical philosophy of set theory that's happened in the last 20 years or so.
- What I want to do in this talk is present the other objective:

> **Main Claim.** We should view the various views about the 'ontology' of set theory (e.g. universism, multiversism) as attempts to engineer new concepts and establish their uptake.

- **Note:** The draft will be ready soon and the book will be submitted at the end of September. If anyone wants to see the draft please let me know—the book needs a few rounds of hard sparring before it's fight ready.

Contents

1 Concepts, conceptions, and conceptual engineering

- This conference concerns **conceptual engineering** and the **concept of collection**.
- Helpful here will be a distinction pointed to by [Incurvati, 2020] between *concepts* and *conceptions*.
- To see this idea, let's start with the concept of FAIRNESS.

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> **Effort/Outcome.** Suppose someone is going to be rewarded over someone else by their company for their work on a case (let's say there was a good outcome but the person did not put much work in). Jane and Susan disagree over whether this decision is fair, Susan thinks companies should reward employees on the basis of outcomes, whereas as Jane thinks that companies should reward employees on the basis of effort.

- Here, Jane and Susan may both have the concept of FAIRNESS.
- But they have different *conceptions* of FAIRNESS, Susan has the <u>fairness-by-outcome conception</u> of FAIRNESS, whereas Jane has the <u>fairness-by-effort conception</u> of FAIRNESS.
- Incurvati then uses this distinction to talk about the concept SET and different conceptions of SET (e.g. the <u>iterative conception</u> of SET).
- **Note:** I'll distinguish between conceptions of concepts by the use of **underline** for conceptions and **caps** for concepts.
- Notice, however, that the notion of concept and conception can be **relativised**. e.g. I can talk about the **concept** of FAIRNESS-BY-OUTCOME, where someone can possess this or lack it. e.g.

> Let's suppose that Mar is rewarded for making their company a good deal of money, but had to do something socially problematic in the course of doing so. Anwar and Bo disagree on whether this is fair. Both Anwar and Bo use the concept of FAIRNESS-BY-OUTCOME, but Bo has the conception that FAIRNESS-BY-OUTCOME should be understood in terms of making the company money, whereas Anwar thinks it is the benefit to society as a whole that is important.

- **Note:** Even Jane, who has the <u>fairness-by-effort conception</u> of FAIRNESS can engage in this discussion.
- She **possesses** the concept of FAIRNESS-BY-OUTCOME she just thinks adopting this concept is the **wrong conception** of FAIRNESS.
- She can discuss this concept **even if** she thinks that the concept is **inconsistent**—cf. [Scharp, 2013]'s example of RABLE (where RABLE applies to x if x is a table, and disapplies to x if x is red, cf. also the concept MASS).
- We can speak of **constitutive principles** for **both** concepts and conceptions.
- A rule or condition is **constitutive** for a concept when they (partly) determine the meaning of the concept and facts about conceptual identity (assuming that there is such a thing).
- Constitutive principles are important as a **lack of agreement** on constitutive principles between speakers can work as an interpretive 'red flag' that speakers do not **mean the same thing** by the use of their words.[1]

> **Conceptual engineering** is (roughly speaking) the field that concerns itself with the evaluation, design, and implementation of our concepts and conceptions.[a]
>
> [a]This definition is adapted from [Chalmers, 2020].

[1]See here [Scharp, 2013], p. 50.

2 Conceptual engineering has happened: The iterative and stratified conceptions of set

- **What is a set**? (i.e. what are the **constitutive principles** for SET?)

  Definition 1. (Informal) A *set* is a kind of *collection* that is:

  - **Extensional.** Sets with different members are non-identical, and sets with the same members are identical.
  - **Objectual.** Sets are *objects* over and above their elements.

- I am going to claim that our concept SET changes (or at least, the reference of our terms "set" changes, and our framework of concepts and conceptions as it pertains to our use of that word changes).
- It will be helpful to contrast this with it's polar opposite:

  Strong Realism. Attached to our talk of "sets" is a unique concept SET which is captured by the iterative conception of SET.

- Let's note first that we have already had some some concept/conception **shift** by adopting SET **at all**.
- We have the set-theoretic conception of COLLECTION.
- Collections can be both **non-objectual** (e.g. pluralities) or **intensional** (e.g. property extensions).
- So moving to SET is already a substantial piece of engineering!
- I want to consider how SET gets engineered.
- Thankfully [Incurvati, 2020] has already done much of this.
- We started with the naive conception of SET.
- As part of the naive conception we have the idea that the **naive comprehension schema** is true: $\exists x \forall y \big(y \in x \leftrightarrow \phi(x)\big)$
- But as we know this leads to contradiction via Russell's paradox and the condition $\phi(x) =_{df} x \notin x$.
- A diagnosis from [Incurvati, 2020], Russell's Paradox results from the way that the Naive Comprehension Schema allows for the following two constitutive principles for SET:
  - **Universality.** A concept C is universal iff there exists a set of all the things falling under C. ([Incurvati, 2020], p. 27)
  - **Indefinite extensibility.** A concept C is indefinitely extensible iff whenever we succeed in defining a set u of objects falling under C, there is an operation which, given u, produces an object falling under C but not belonging to u. ([Incurvati, 2020], p. 27)
- Two conceptions of set that have arisen in response:
- The iterative conception of SET. Sets are formed in stages, starting with the empty set (or suitable Urelemente) taking powersets at successor stages and unions at limits.
- The stratified conception of SET holds that sets are the kinds of things that are given by definitions respecting **typing restrictions**.
- (**Note:** [Incurvati, 2020] thinks that the stratified conception of SET is really a conception of OBJECTIFIED PROPERTY. I'll put this to one side.)

- The iterative conception and stratified conception each give up a separate constitutive principles of the naive conception.
- The iterative conception gives up **universality** and the naive conception gives up **indefinite extendibility**.
- Compare this, for example, with [Scharp, 2013]'s ascending conception and descending conception of TRUTH.
- [Incurvati, 2020] suggests that we pursue a strategy of **inference to the best conception**—compare the various conceptions of SET and their theoretical virtues.
- Part of these could involve e.g.
  - Explanation of the paradoxes.
  - Motivation of a nice theory of sets.
  - Respecting foundational constraints (e.g. **Generous Arena**).
- Let's note that the iterative conception of SET is probably consistent.
- On the one hand, it's a nice motivation for the axioms of ZFC...
- But on the other it is a **mathematical fact of life**:
- Defining the V_α in the usual way, we have ZFC $\vdash \forall x \exists \alpha (x \in V_\alpha)$

3 The absoluteness conception of maximal iterative set

- And as we know, **no** controversy in the philosophy of set theory ever arose, and the mathematicians lived **happily ever after**.
- Sadly, this **isn't** the story.
- The iterative conception of SET is **consistent**, but **defective**.
- Note that **consistency** is a **low**(ish) bar to clear.
- We also want set theory to provide a **Theory of Infinity**, to tell us **what sizes of sets** are out there and what the **relationships** between them are.
- (**Note:** There are many other foundational goals that I discuss in the book, most of which I take from [Maddy, 2017] and [Maddy, 2019].)
- The iterative conception of SET fails **spectacularly badly here**.
- Do **large cardinals exist**?
- What is the behaviour of the **continuum function**?
- The iterative conception tells us **almost nothing** here.
- One thing that has happened is that many set theorists have moved to the maximal conception of ITERATIVE SET.
- The maximal conception of ITERATIVE SET adds the constitutive principle that there should be **as many sets as possible**.
- **Problem:** This is also a pretty **uninformative** constitutive principle.
- There are **all sorts** of maximality principles, and many disagree with each other (see [Incurvati, 2017] for a survey).

- For a simple example, CH can be seen as maximising (lots of **reals**!) and so can ¬CH (lots of **functions**!).
- So we need to **sharpen** further.
- There's **lots** of ways we could go here, and things now start to get a little **stipulative**, but we can move to:
- The absoluteness conception of MAXIMAL ITERATIVE SET holds that:
  - **Capture.** The sets all exist within a single universe.
  - **Absoluteness.** If there **could** be a set such that ϕ then there **is** a set such that ϕ.
- Formally: $\Diamond\big(\exists x\phi(x)\big) \rightarrow \exists y\phi(y)$
- OK what does it mean for a set to be **possible** here.
- I'll take this to mean: Could be obtained either by **adding ranks** or by **moving to a forcing extension**.
- Accordingly, we have:
  - **Height absoluteness.** $\Diamond_h\big(\exists x\phi(x)\big) \rightarrow \exists y\phi(y)$
  - **Width absoluteness.** $\Diamond_w\big(\exists x\phi(x)\big) \rightarrow \exists y\phi(y)$
- **Note:** Don't **freak out**, this can all be coded up! (cf. [Antos et al., 2021]).
- Let's **restrict** to Σ_1-sentences, since we can clearly run into issues with Σ_2-sentences (e.g. both CH and ¬CH are Σ_2).
- This looks **promising** (OK, you can probably see where I'm going, but humour me).
- Presumably it's possible for there to be uncountable cardinals and inaccessible cardinals, by making Ord into a set.
- So we get **large cardinals** (given suitable possibility axioms).
- We also get resolutions to CH (in the negative) via **bounded forcing axioms** (e.g. BPFA) that have absoluteness characterisations.
- (e.g. BPFA can be stated as the claim that if ϕ is a Σ_1 sentence with parameters from $\mathcal{P}(\omega_1)$, then if ϕ holds in a forcing extension obtained by proper forcing, then ϕ holds.)
- So we seem to be making **some** progress.

4 A 'new' kind of paradox

- Unfortunately the absoluteness conception of MAXIMAL ITERATIVE SET is **inconsistent**.
- **The Cohen-Scott Paradox** begins by observing that by **Height Absoluteness** in combination with **Capture**, there should be lots of uncountable sets and large cardinals.
- But also, by **Width Absoluteness** and **Capture**, any particular set x you consider should be countable.
- Take any uncountable set x.
- By forcing, there is a bijection $f : x \rightarrow \omega$ in a forcing extension.
- By width absoluteness there is such a bijection $f : x \rightarrow \omega$.
- Contradiction!
- **OK:** What has gone wrong here?

- On the one hand our desire for **uncountable** sets and **height absoluteness** pushes us to say that there are lots of large cardinals (any uncountable cardinal is large for me).
- On the other hand **width absoluteness** just wants to **kill off** the idea that cardinals have closure properties.
- **Capture** ensures that all this happens in the same universe.
- (**Note:** No-one really runs into this paradox in formal work quite like we did with **Russell**. Set theorists are not dummies, and they see this problem a mile off. **But** it represents the two directions that the **set-theoretic community as a whole** seems to be pulled in.)

5 Contemporary engineering

- Normally, we see **multiversism** and **universism** as claims about **ontology**—there **is** (**not**) a set-theoretic universe that is thus and so.
- But here we can see the different views as proposing different **conceptions** that give up one or more of **Width Absoluteness**, **Height Absoluteness**, and **Capture**.
- **Option 1.** High universist conception of MAXIMAL ITERATIVE SET. Keep **Capture**, keep **Height Absoluteness**, curtail **Width Absoluteness** (by rejecting that forcing extensions are really possible). (Bagaria, Woodin)
- **Option 2.** Wide universist conception of MAXIMAL ITERATIVE SET. Keep **Capture**, keep **Width Absoluteness**, curtail **Height Absoluteness** (again, via claims about what's possible). ([Barton and Friedman, S])
- **Option 3.** High multiversist conception of MAXIMAL ITERATIVE SET. Reject **Capture**, keep **Height Absoluteness**, reinterpret **Width Absoluteness** ([Steel, 2014])
- **Option 4.** Wide multiversist conception of MAXIMAL ITERATIVE SET. Reject **Capture**, curtail/reinterpret **Height Absoluteness**, keep **Width Absoluteness** ([Arrigoni and Friedman, 2013]).
- **Option 5.** Schematic multiversist conception of MAXIMAL ITERATIVE SET. Reject **Capture**, reinterpret **Height Absoluteness** and **Width Absoluteness** ([Hamkins, 2012], [Scambler, 2021], [Builes and Wilson, 2022]).
- Each conception is (probably) **consistent**, witnessed by the **models** we have for each.
- We can continue our strategy of pursuing **inference to the best conception**.
- Each resolves the defect with respect to **Theory of Infinity** to a greater/lesser extent.
- But there are other trade offs to be made (e.g. with respect to **Generous Arena**).
- I think there is a case to be made that some of these conceptions are reaching the status of concepts for certain researchers, and this can explain some of the misunderstanding.

6 Conclusions and open questions

- I think it's clear that we are at a **conceptual crossroads** with set theory, and these proposal about '**ontology**' can be seen as proposals for how to **move forward** with our theory of (maximal) sets.
- There is a **plethora** of open questions (as well as **objections**) and I'll let them come up in discussion!

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