Countabilism and Maximality (or 'Some Systems of Set Theory on which Every Set Is Countable')

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Introduction and Motivation

- ZFC-based set theory is a very useful mathematical theory.

- Consider the following desiderata on a foundation (taken from [Maddy, 2019]):

Generous Arena. Find *representatives* for our usual mathematical structures (e.g. $\mathbb{N}$, $\mathbb{R}$) in our theory of sets.

Shared Standard. Provide a standard of correctness for proof in mathematics.

Metamathematical Corral. Provide a theory in which metamathematical investigations of relative provability and consistency strengths can be conducted.

Risk Assessment. Provide a degree of confidence in theories commensurate with their large cardinal strength.

To this, we might add:

Motivational Challenge. Motivate a theory with a substantial degree of large cardinal strength on the basis of an account of the global nature of the universe.

- ZFC, combined with the iterative conception (and reflection principles) performs *wonderfully* here. Part of what does the work is:

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Maximality. The universe of sets should be as *large* as possible.

- (Of course this is vague, we're just looking at *motivation* rather than *justification*.)

A Motivating Puzzle: Cohen's Paradox

Observation A. (Cantor's Theorem) The Powerset Axiom implies that there are uncountable sets.

Observation B. Given any model of set theory $\mathfrak{M} = (M, E)$, and any $\mathfrak{M}$-cardinal κ, there is forcing partial order $Col(\omega, \kappa)$ which forces κ to be countable in the extension.

Observation C. Given any model of set theory $\mathfrak{M} = (M, E)$, and any $\mathfrak{M}$-cardinal κ, there is a forcing $Add(\mathcal{P}^{\mathfrak{M}}(\omega, \kappa^+))$ that pushes the value of the continuum above κ in the extension.

The Cohen-Scott Paradox.

- We think that there are uncountable sets (in particular the set of all real numbers) by Cantor's Theorem.

- But by Observation B, we (in some sense) 'could' collapse any set x to the countable by adding a surjection $f : \omega \to x$...

- ...and in particular (by Observation C) we 'could' make the reals bigger than x.

- On the one hand we think that the V should contain uncountable sets, but on the other hand, if the universe does contain uncountable sets it appears to be missing all sorts of collapsing generics for partial orders, and in particular the reals seem smaller than they might have been.

- Of course the standard response to the Cohen-Scott Paradox is that it shows that various kinds of models are *unintended* in some sense—for example they may be Boolean-valued or countable.

- The sense in which the reals 'might' have been larger than a particular cardinal is only with respect to some unintended interpretation.

- Note, however, that this is just *one* of the available responses. A different approach would be to say that it is the interpretation of Cantor's Theorem that is at fault.

- Another possible option: Deny the Powerset Axiom and hold that every set is countable and the continuum is a proper class.

This isn't such a heresy:

> I see that there are any number of contradictory set theories, all extending the Zermelo-Fraenkel axioms: but the

models are all just models of the first-order axioms, and first-order logic is weak. I still feel that it ought to be possible to have strong axioms, which would generate these types of models as submodels of the universe, but where the universe can be thought of as something absolute. Perhaps we would be pushed in the end to say that all sets are countable (and that the continuum is not even a set) when at last all cardinals are absolutely destroyed. ([Scott, 1977], p. xv)

Challenge. Can we put the countabilist (the believer that every set is countable) in a similar position to the ZFC-uncountabilist with respect to the foundational constraints (including the **Motivational Challenge** and **Maximality**)?

We'll argue: **Certainly we can get close.** (...and we get some interesting results along the way, one of our principles will imply that $0^\sharp$ exists and is consistent relative to ZFC + PD).

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- §2 Brute forcing large cardinal strength.
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- §4 The Axiom of Set-Generic Absoluteness
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- §6 The Ordinal Inner Model Hypothesis
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1 Removing Powerset

-One issue that needs to be dealt with briefly before we get into the main part of our proposal is: What do we take the countabilist's base theory to be?

-A natural answer: It is the theory ZFC with the Powerset Axiom removed with "Every set is countable" added, call this latter axiom Count.

-Problem: [Zarach, 1996] and [Gitman et al., 2011] show that various equivalences one normally has in the presence of the Powerset Axiom disappear once it is removed.

-In particular, simply deleting the Powerset Axiom and keeping Replacement does not preserve Collection, and various versions of the Axiom of Choice become non-equivalent.

-We therefore need to set up what we mean by various theories lacking the Powerset Axiom:

Definition 1. We distinguish between the following theories:

(1.) $\mathsf{ZFC}-$ is ZFC with the Powerset Axiom Removed and AC formulated as the claim that every set can be well-ordered.

(2.) ZFC^- is $\mathsf{ZFC}-$ with the Collection and Separation Schema substituted for the Replacement Scheme.

(3.) ZFC^-_{Ref} is ZFC^- with the following schematic reflection principle added (for any ϕ in the language of set theory):

$$\forall x \exists A (x \in A \wedge \text{“}A \text{ is transitive”} \wedge \phi \leftrightarrow \phi^A)$$

i.e. for any set x there is a transitive set A such that $x \in A$ and ϕ is absolute between A and the universe. We will refer to this principle as (the) *First-Order Reflection (Principle)*.

(4.) By $\mathsf{NBG}-$, NBG^-, and NBG^-_{Ref} we mean the corresponding versions of NBG, with two sorts of variables and any corresponding schema replaced by single second-order (predicative) axioms.

- It is known that the three theories $\mathsf{ZFC}-$, ZFC^-, and ZFC^-_{Ref} are distinct ([Zarach, 1996], [Gitman et al., 2011]) and that there are models of ZFC^- + Count in which reflection fails ([Friedman et al., F]).

- First-order reflection is equivalent (modulo ZFC^-) to:

Definition 2. (ZFC^-) The *Dependent Choice Scheme* (we will also use the terms 'DC-Scheme' or DCS as appropriate) is the scheme of assertions claiming that for each formula $\phi(x, y, z)$ and parameter a, if for every x there is a y such that $\phi(x, y, a)$ holds, then there is an ω-sequence $\langle x_n | n \in \omega \rangle$ such that for all n, $\phi(x_n, x_{n+1}, a)$ holds. (i.e. If a definable relation has no terminal nodes, we can make ω-many dependent choices on its basis.)

2 The Brute Force Strategy

- The first point to be made is that there *are* statements of set theory concerning second-order arithmetic (and hence ZFC^- + Count) that yield substantial large cardinal strength.

- The core observation is that much recent work in set theory has involved building inner models for large cardinals from principles of second-order arithmetic.

- One that will interest us here is “$0^\sharp$ exists”.

- Some care here is needed as some formulations won't work (e.g. "Every uncountable cardinal is indiscernible in L" is clearly useless in this context.)

- We'll use the following:

Definition 3. ($\mathsf{ZFC}^-/\mathsf{NBG}^-$)"$0^\sharp$ exists" will be taken to mean that there is a definable club of L-indiscernibles.

- Note that this isn't *essentially* higher-order: Silver's work shows that if there is a definable club of L-indiscernibles then there is a unique Δ_2-definable (with parameters) such club, and of course Δ_2-definability *is* definable.

- But now we can note some facts:

Fact 4. NBG^-+*"$0^\sharp$ exists" implies that there is an inner model (i.e. a transitive model containing all the ordinals) for* ZFC+*"There exists a proper class of inaccessible cardinals".*

Proof Sketch. Every Silver indiscernible is L-inaccessible, and so the existence of $0^\sharp$ implies the existence of a proper class of inaccessible cardinals in L. Moreover, these indiscernibles are elementary in L and so $L \models \mathsf{ZFC}$. □

- We can go further:

- (Woodin) With Projective Determinacy one gets inner models with Woodin cardinals (in particular n-many for every $n \in \mathbb{N}$).

- (Krapf) From $\underset{\sim}{\Pi}^1_1$-Determinacy and the $\underset{\sim}{\Pi}^1_2$-Perfect Set Property one can obtain inner models of ZFC+"Every set of ordinals has a sharp".

- So we *can* get large cardinal strength from seemingly natural principles under ZFC^- + Count.

- In this context, whilst some inner models have ZFC (and indeed much more) they are impoverished with respect to the functions they can see (in particular they are blind to all sorts of collapsing functions).

- But are these principles *well-motivated* (in line with the **Motivational Challenge** and **Maximality**)?

> "Is PD true? It is certainly not self-evident." ([Martin, 1977], p. 813)

3 Forcing Saturated Set Theory

- Let's see if we can do better.

- Our suggestion is to view the maximality of the universe via different kinds of *saturation* under possible sets.

- We'll first look at a *forcing axiom* type strategy (that is *weak*, it fails to break $V = L$ and is consistent relative to ZFC$^-$).

- Normally in the ZFC context, when formulating a forcing axiom (asserting that for certain partial orders and families of dense sets there are generics) we have to be *careful*, go too far and you get contradiction (e.g. $MA(\mathfrak{c})$).

- But we haven't committed to the existence of uncountable cardinals, so we can formulate:

Definition 5. (ZFC$^-$) *The Forcing Saturation Axiom* (or FSA). If $\mathbb{P}$ is a forcing poset, and $\mathcal{D}$ is a set-sized family of dense sets, then there is a filter $G \subseteq \mathbb{P}$ intersecting every member of $\mathcal{D}$. The theory of *Forcing Saturated Set Theory* or FSST comprises ZFC$^-$ + FSA.

FSST implies that every set is countable. However, it is also weak, as is shown by the following fact:

Fact 6. *(*ZFC$^-$*)* FSST *is equivalent to the theory* ZFC$^-$ + Count*.*

Proof Sketch. (1.) FSST $\Rightarrow$ ZFC$^-$+"Every set is countable".

- For any ordinal α, use $Col(\omega, \alpha)$ (get this from definable powerset) to get a collapse from α onto ω. (You need to define the right dense sets etc. but it goes through pretty easily.)

(2.) ZFC$^-$+"Every set is countable" $\Rightarrow$ FSST.

- Let $\mathbb{P}$ be a forcing poset and $\mathcal{D}$ be a family of dense subsets of $\mathbb{P}$.

- Since every set is countable, we can enumerate $\mathcal{D}$ in order-type ω.

- So, without loss of generality, $\mathcal{D} = \langle D_n | n \in \omega \rangle$. Since every set is countable, $\mathbb{P}$ can also be enumerated in order-type ω, let 'f' denote the relevant enumerating function.

- We can then define via recursion (and using the parameter f) the following function π from $\mathcal{D}$ to $\mathbb{P}$:

$$\pi(D_0) = \text{“The } f\text{-least } p \in D_0\text{”}$$

$$\pi(D_{n+1}) = \text{“The } f\text{-least } p \in D_{n+1} \text{ such that } p \leqslant_{\mathbb{P}} \pi(D_n)\text{”}$$

- Effectively π successively picks elements of each member of $\mathcal{D}$, ensuring that we always go below our previous pick in the forcing order.

- $ran(\pi)$ then generates the relevant generic. □

By Fact 6 we have the immediate:

Corollary 7. FSST *is consistent relative to the theory* ZFC$^-$*.*

Corollary 8. FSST *is consistent with* $V = L$*.*

- Thus whilst FSST does imply countabilism through some sort of saturation idea, it fails to break $V = L$ and is weak.

- We should not necessarily view this fact as a deathblow to FSST, however.

- We might rather view FSST as an initial stepping stone to stronger theories, much like how ZFC is consistent with $V = L$ but can be strengthened using large cardinals to theories that break $V = L$.

4 The Axiom of Set-Generic Absoluteness

- OK so postulating generics wasn't quite strong enough.

- It also maybe wasn't *completely* in line with the **Motivational Challenge**.

- We'll use the idea of *absoluteness in width* to generate some stronger principles.

- Note that the usual reflection principles are essentially *rank* absoluteness principles.

- In this section we'll break $V = L$, but not get any additional large cardinal strength.

- Ideas of width absoluteness principles already exist in the literature on bounded forcing axioms:

Definition 9. [Bagaria, 2000] *Absolute*-BPFA. We say that $V \models$ ZFC satisfies *Absolute*-BPFA iff whenever ϕ is a $\Sigma_1(\mathcal{P}(\omega_1))$ formula, if ϕ holds a forcing extension $V[G]$ obtained by proper forcing, then ϕ holds in V.

> In the case of MA and some weaker forms of PFA and MM, some justification for their being taken as true axioms is based on the fact that they are equivalent to principles of generic absoluteness. That is, they assert, under certain restrictions that are necessary to avoid inconsistency, that everything that might exist, does exist. ([Bagaria, 2008], pp. 319–320)

We take this to the ZFC$^-$ context:

Definition 10. (ZFC$^-$) We say that V, a model of ZFC$^-$, satisfies the *Weak Axiom of Set-Generic Absoluteness* (WASGA) iff whenever $\phi(\vec{a})$ is a Σ_1-formula in the language of set theory in the parameters $\vec{a} \in V$, if $\mathbb{P} \in V$ is a forcing partial order, G is V-generic in the sense that it intersects **every** dense set in V, and $\phi(\vec{a})$ holds in $V[G] \models$ ZFC$^-$, then $\phi(\vec{a})$ holds in V.

Unfortunately:

Fact 11. *(*ZFC$^-$*) The* WASGA, FSA, *and* Count *are equivalent (modulo* ZFC$^-$*).*

Proof Sketch.

- WASGA $\Rightarrow$ Count is almost immediate.

- Count $\Rightarrow$ WASGA follows from Lévy Absoluteness which tells us that if a Σ_1-formula with real parameters holds in an outer model of ZFC$^-$ then it holds in V.

- So if V satisfies ZFC$^-$ + Count then WASGA will hold for Σ_1-formulas (since under Count *every* set is coded by a real). $\square$

- Well, let's go wild and allow *any* sentence:

Definition 12. (ZFC$^-$) We say that V, a model of ZFC$^-$, satisfies the *Axiom of Set-Generic Absoluteness* (or ASGA) iff whenever $\phi(\vec{a})$ is a sentence in the language of set theory *in the parameters* $\vec{a} \in V$, if $\mathbb{P} \in V$ is a forcing partial order, G is V-generic in the sense that it intersects *every* dense set in V, and $\phi(\vec{a})$ holds in $V[G] \models$ ZFC$^-$, then $\phi(\vec{a})$ holds in V.

OK, so obviously:

Fact 13. *Over the theory* ZFC$^-$ *the* ASGA *implies the* FSA*.*

But:

Fact 14. ZFC$^-$ + ASGA *implies that* $V \neq L$*.*

- We might worry about consistency.

- Normally generic absoluteness says that if there is a set in a forcing extension that satisfies an absolute property then there is such a set in the ground model.

- Typically the underlying absolute property is Δ_0, hence many generic absoluteness axioms (e.g. Absolute-BPFA) postulate Σ_1-absoluteness.

- Somewhat surprisingly, the ASGA is actually very *weak* in terms of consistency strength:

Fact 15. ZFC$^-$ + ASGA *is consistent relative to* ZFC$^-$*.*

Proof Sketch.
- We'll do it in ZFC first, but it's pretty easy to reduce the strength.

- We begin with a model M of ZFC

- Force using an $\aleph_1$-product of Cohen forcings with finite support (call this forcing $\mathbb{P}$), to form an extension $\mathfrak{M}[G]$.

- We claim that $H(\omega_1)^{\mathfrak{M}[G]}$ satisfies ZFC$^-$ + ASGA.

- The fact that ZFC$^-$ holds is immediate, since the $H(\omega_1)$ of any model of ZFC satisfies FSST.

- It just remains to argue that $H(\omega_1)^{\mathfrak{M}[G]}$ satisfies the ASGA.

- Any finite sequence of parameters $\vec{a}$ from $H(\omega_1)^{\mathfrak{M}[G]}$ appears at some stage of the iteration.

- In other words, if we let G_α be the first α-many Cohen reals added by G, then $\vec{a}$ appears in $V[G_\alpha]$.

- Since $\vec{a}$ is hereditarily countable, it can be coded by some real r.

- Moreover, r must belong to $V[G_\alpha]$ for some countable α (by the ccc. and tracing the names through $\mathbb{P}_\alpha$).

- In other words, any real r added by G is already added for some G_α, for countable α.

- Letting $G_{\alpha\rightsquigarrow}$ be the Cohen reals added after G_α by $\mathbb{P}$, we can then view $H(\omega_1)^{\mathfrak{M}[G]}$ as $H(\omega_1)^{\mathfrak{M}}[G_\alpha][G_{\alpha\rightsquigarrow}]$, where $G_{\alpha\rightsquigarrow}$ is $H(\omega_1)^{\mathfrak{M}}[G_\alpha]$-generic for the ω_1-many Cohen forcings after the α^{th} stage of the iteration given by $\mathbb{P}$.

- Now suppose that there is a countable forcing $\mathbb{Q}$ in $H(\omega_1)^{\mathfrak{M}[G]} = H(\omega_1)^{\mathfrak{M}}[G_\alpha][G_{\alpha\rightsquigarrow}]$, and generic $G_{\mathbb{Q}}$ such that $H(\omega_1)^{\mathfrak{M}[G]}[G_{\mathbb{Q}}] \models \phi(\vec{a})$ where $\vec{a} \in H(\omega_1)^{\mathfrak{M}[G]}$.

- To show that the ASGA is satisfied by $H(\omega_1)^{\mathfrak{M}[G]}$, we just have to show that $H(\omega_1)^{\mathfrak{M}[G]} \models \phi(\vec{a})$.

- Since $G_{\mathbb{Q}}$ is generic over $H(\omega_1)^{\mathfrak{M}[G]}$ for a countable forcing (i.e. $\mathbb{Q}$), we can assume without loss of generality that $G_{\mathbb{Q}}$ is generic for Cohen forcing, since Cohen forcing is the only countable forcing up to forcing-equivalence.

- Thus, since $H(\omega_1)^{\mathfrak{M}[G]} = H(\omega_1)^{\mathfrak{M}}[G_\alpha][G_{\alpha\rightsquigarrow}]$, we know that $H(\omega_1)^{\mathfrak{M}[G]}[G_{\mathbb{Q}}] = H(\omega_1)^{\mathfrak{M}}[G_\alpha][G_{\alpha\rightsquigarrow}][G_{\mathbb{Q}}]$, and hence that $\phi(\vec{a})$ becomes true after forcing with the finite support product over $H(\omega_1)^{\mathfrak{M}[G_\alpha]} = H(\omega_1)^{\mathfrak{M}}[G_\alpha]$, adding $G_{\alpha\rightsquigarrow}$ and $G_{\mathbb{Q}}$, i.e. adding (ω_1+1)-many Cohen reals (which is just ω_1-many Cohen reals).

- It follows (using the homogeneity of Cohen forcing) that $H(\omega_1)^{\mathfrak{M}}[G_\alpha][G_{\alpha\rightsquigarrow}] = H(\omega_1)^{\mathfrak{M}}[G] \models \phi(\vec{a})$, as required.

- To reduce the strength of our initial assumption to ZFC^-, we cannot simply use an $\aleph_1$-product of Cohen forcings with finite support, since we have no guarantee that $\aleph_1$-exists.

- Supposing that it does not, we can force with the finite support product of Ord-many Cohen forcings (i.e. Ord now plays the role of ω_1).

- This is a class forcing, but it is ZFC^- preserving and we can run the same argument as above. $\square$

- Thus we see (surprisingly given the strength of the generic absoluteness postulated) that the ASGA is consistent relative to ZFC^-.

- The reason for consistency (as shown by the previous proof) is extremely special to the countabilist context: If every set is countable then all set-forcings are equivalent to Cohen forcing.

- OK, so: The ASGA is interesting, breaks $V = L$, but is ultimately *weak* (and thus fails the **Motivational Challenge**).

5 The Extreme Inner Model Hypothesis

- We would now like to try developing absoluteness principles that imply that every set is countable, but have greater strength than either the FSA or ASGA.

- In this section, we blow everything up.

- A natural point of attack is the extensions allowed (so far we've only allowed set forcing).

- This has been examined in the ZFC-context:

Definition 16. [Friedman, 2006] Let ϕ be a parameter-free first-order sentence. The *Inner Model Hypothesis* (or IMH) states that if ϕ is true in an inner model of an outer model of V, then ϕ is already true in an inner model of V.

- The IMH is *strong* (measurable cardinals of arbitrarily high Mitchell-order in inner models), *consistent* (relative to a Woodin with an inaccessible above), and *kills large cardinals* (no worldly cardinals in V).

- We therefore formulate:

Definition 17. (NBG^-_{Ref}) *Extreme Inner Model Hypotheses.* Let $\phi(\bar{a})$ be a first-order sentence with parameters $\bar{a} \in V$. The *Extreme Inner Model Hypothesis for* $\mathsf{ZFC}-$ (respectively ZFC^-, ZFC^-_{Ref}) or $\mathsf{EIMH}-$ (respectively EIMH^-, EIMH^-_{Ref}) states that if $\phi(\bar{a})$ is true in an inner model of an outer model $(V^*, \in^*,) \models \mathsf{ZFC}^-$ of V, then $\phi(\bar{a})$ is already true in an inner model of V.

- Before we proceed, a remark concerning formalisation of these principles is in order.

- First, given some appropriate theory T, the EIMH for T admits of various formulations according to how we wish to code the notion of 'outer model'.

- For everything we say here, we'll only need definable class forcings which are pretame and have a definable forcing relation.

- One can formulate the principles to insist on NBG^- in the outer models with a bit of extra metamathematical work. (But you do *not* need resources beyond NBG^- to formulate it for pretame class forcing.)

- Since our focus is on just presenting the mathematical results in this talk, we'll just work over a suitable countable transitive model and suppress the metamathematics. See the paper (going up in the next fortnight or so) for the details.

Fact 18. $\mathsf{NBG}^- + \mathsf{EIMH}^-$ *implies* Count *(equivalently the* FSA *and the* WASGA*).*

Fact 19. $\mathsf{NBG}^- + \mathsf{EIMH}^-$ *implies that* $V \neq L$*.*

Unfortunately:

Theorem 20. NBG^-_{Ref} + EIMH^-_{Ref} *is inconsistent.*

Proof Sketch.

- We will show that there is no transitive model of NBG^-_{Ref} + EIMH^-_{Ref}.

- The proof will in fact show that NBG^-_{Ref} + EIMH^-_{Ref} proves there is no transitive model of a particular finite subtheory T of NBG^-_{Ref} + EIMH^-_{Ref}.

- This is fine: NBG^-_{Ref} + EIMH^-_{Ref} proves that T has a transitive model (by Reflection/DCS) we infer the inconsistency of NBG^-_{Ref} + EIMH^-_{Ref}.

- Suppose that V is a transitive model of NBG^-_{Ref} + EIMH^-_{Ref} of ordinal height α, we may assume that V is countable.

- Note that V satisfies Count (by the EIMH^-_{Ref}).

- Now as in the proof of Theorem 3.8 (using reshaping and almost disjoint forcing) of [Antos and Friedman, 2017], we can produce an outer model of V satisfying ZFC^-_{Ref} which is of the form $L_\alpha[r_0]$ for some real r_0.

- And as in the proof of Theorem 4.1 of [Antos and Friedman, 2017] (using a small modification of the reshaping argument) we can enlarge further to a model of ZFC^-_{Ref} of the form $L_\alpha[r]$ for some real r such that for every ordinal $\beta < \alpha$, $L_\beta[r]$ fails to satisfy Collection.

- Applying the EIMH^-_{Ref}, there is such a real, which we denote by r', in the original model V.

- For each $\beta < \alpha$ let $f(\beta)$ be the least n so that Σ_n-Collection fails in $L_\beta[r']$.

- For each n we can force over V to add a club C_n consisting of ordinals $\beta < \alpha$ such that $f(\beta)$ is at least n. (This is in Proposition 3.5 of [Friedman, 2000].)

- And again as in the proof of Theorem 3.8 of [Antos and Friedman, 2017], we can with further forcing add a real s_n which codes C_n.

- By the EIMH^-_{Ref} in V, there are such reals s_n in V, coding corresponding clubs C_n.

- But taking some β belonging to the intersection of all the various C_n, we have that $f(\beta)$ is at least n for each n, a contradiction. $\square$

- OK, so it looks like the EIMH idea takes things too far.

- This isn't a deathblow to the countabilist, the same is true for the uncountabilist (almost any axiom proposed can be extended to inconsistency, including reflection).

6 Ordinal Inner Model Hypotheses

- We are now in a position where we would like to weaken the EIMH but still go beyond the ASGA.

- A natural choice here is to restrict the parameters allowed:

Definition 21. (NBG^-) *Ordinal Inner Model Hypotheses.* Let $\phi(\vec{a})$ be a first-order sentence with **ordinal** parameters $\vec{a}$. The *Ordinal Inner Model Hypothesis for* T (or $\mathsf{OIMH}^{\mathsf{T}}$) states that if $\phi(\vec{a})$ is true in an inner model of an outer model of V satisfying T, then $\phi(\vec{a})$ is already true in an inner model of V. We will refer to the $\mathsf{OIMH}^{\mathsf{T}}$ for $\mathsf{T} = \mathsf{ZFC}-$, $\mathsf{T} = \mathsf{ZFC}^-$, and $\mathsf{T} = \mathsf{ZFC}^-_{Ref}$ as the $\mathsf{OIMH}-$, OIMH^-, and OIMH^-_{Ref} respectively.

- Clearly the OIMH-style principles have all the nice countabilist things we wanted from earlier (we get Count and break $V = L$). But we also have:

Theorem 22. NBG^-_{Ref} + OIMH^-_{Ref} *is consistent relative to the theory* ZFC + PD.

Proof Sketch.
- The strategy of the proof is to work in a model of ZFC + PD and use the structure of Turing degrees given by PD to ensure that we can find models with the right behaviour.

- For any set x of ordinals let $M(x)$ denote the least transitive model of ZFC^-_{Ref} containing x (such a model is of the form $L_\beta[x]$ for some β and satisfies the DC-scheme).

- We now define a function that will be useful in finding models with the same theory (for extracting the inner models required for the OIMH^-_{Ref} later).

- For each countable ordinal α let $f(\alpha)$ be a real r_α such that α is countable in $M(r_\alpha)$ and for all y in which r_α is recursive we have that $M(y)$ has the same theory with parameter α as $M(r_\alpha)$.

- We use PD to check that f is well-defined: First note that PD implies (by Martin's Cone Lemma in [Martin, 1968]) that any projective set of reals closed under Turing equivalence either contains or is disjoint from a Turing cone.

- Also (in ZFC alone) the intersection of countably many Turing cones contains a Turing cone.

- Now for each sentence ϕ in the language of set theory with parameter α, let X_ϕ be the set of reals x such that α is countable in $M(x)$ and $M(x)$ satisfies ϕ.

- The X_ϕ are closed under Turing-equivalence since if x_0 and x_1 are Turing equivalent then $M(x_0) = M(x_1)$.

- Moreover each X_ϕ is projective (indeed Δ^1_2).

- Next, for each ϕ choose a Turing cone inside either X_ϕ or $X_{\neg\phi}$ and let y be in the intersection of these Turing cones.

- Note that α is countable in $M(y)$ as one of these Turing cones only has reals with α countable. Furthermore, if y is recursive in z it follows that $M(y)$ and $M(z)$ have the same theory with parameter α.

- So f is able to pick a r_α for each countable ordinal α.

- Now let's find our model of the OIMH^-

- Let N^* be a countable elementary submodel of some large $H(\theta)$ with θ regular containing f as an element, and let N be $N^* \cap H(\omega_1)$ (the sets in N^* which are hereditarily countable in N^*).

- Equivalently, N is the $H(\omega_1)$ of the transitive collapse of N^*. As $H(\omega_1)$ satisfies the DCS (given that we are now in ZFC), so does its image under transitive collapse, which is N.

- We now use N to find our model of the OIMH^-_{Ref}. Let β denote the ordinal height of N.

- Similarly to Theorem 20, we use Theorem 3.8 of [Antos and Friedman, 2017] and Theorem 4.1 of [Antos and Friedman, 2017] to force to add a real y so that $N[y] = L_\beta[y]$ is the least model of ZFC^- containing y, i.e. $N[y] = M(y)$.

- We claim that $(M(y), Def(M(y))$ satisfies the OIMH^-_{Ref} (and NBG^-_{Ref}).

- Suppose that ϕ with parameter α (for $\alpha < \beta$) is satisfied in a inner model M_0 of an outer model M of $M(y)$.

- We will find a definable (with parameters) inner model of $M(y)$ satisfying ϕ.

- We first enlarge M (again using the methods of Theorem 20) to a model of the form $M(z)$ (for z a real) in which M is a definable inner model.

- Since M_0 is definable in M and M is definable in $M(z)$, we know that M_0 is a definable inner model of $M(z)$.

- Choose n such that M_0 is a Σ_n-definable inner model of $M(z)$, and let ψ be the sentence: "There is a Σ_n-definable inner model satisfying ϕ". (OK: Here I'm using a step in the NBG^-_{Ref} version of the proof. But we can have a Σ_n ZFC^-_{Ref} formula ξ defining an inner model, with ϕ holding relative to the model defined by ξ.)

- ψ is a sentence with parameter α true in $M(z)$, i.e. ψ belongs to the theory of $M(z)$ with parameter α.

- We now pick a z^* in $M(z)$ that is Turing-above both z and $f(\alpha)$. (For concreteness, we could just let z^* be the join of z and $f(\alpha)$.)

- Now, we know that z^* belongs to $M(z)$ (by assumption) and that $z \in M(z^*)$ (since z is Turing-below z^*).

- We then have that $M(z) = M(z^*)$ since in general $x_0 \in M(x_1)$ implies that $M(x_0) \subseteq M(x_1)$.

- We know that ψ holds in $M(z^*)$ simply because $M(z^*) = M(z)$ and ψ holds in $M(z)$.

- Recalling the definition of $f(\alpha)$, we note that $f(\alpha)$ was chosen specifically so that for any x, $M(f(\alpha))$ and $M(x)$ have the same theory with parameter α for any x that are Turing-above $f(\alpha)$.

- Since z^* is Turing-above $f(\alpha)$, we know that ψ holds in $M(f(\alpha))$.

- We also know that $f(\alpha)$ belongs to $M(y)$ (since $f(\alpha)$ belongs to N), and so we can choose a real y^* in $M(y)$ that is Turing-above both y and $f(\alpha)$.

- Then, as before, $M(y^*) = M(y)$ (since $y^* \in M(y)$ and $y \in M(y^*)$).

- But now, since y^* is Turing-above $f(\alpha)$, $M(y^*)$ has the same theory with parameter α as $M(f(\alpha))$, and so ψ holds in $M(y^*) = M(y)$.

- But ψ exactly says that ϕ holds in a Σ_n-definable inner model, and so ϕ holds in a definable inner model of $M(y)$ as desired. $\square$

-Hooray! The OIMH^-_{Ref} is (probably) consistent.

-But does it have any *large cardinal strength*?

Theorem 23. *(ZFC^-_{Ref}) Suppose that (V, C) satisfies $\mathsf{NBG}^-_{Ref} + \mathsf{OIMH}^-_{Ref}$. Then (V, C) satisfies "$0^\sharp$ exists".*

Proof Sketch.

- Suppose that V satisfies NBG^-_{Ref} + OIMH^-_{Ref}. By preparatory forcing (exactly as in Theorems 20 and 22) we can choose an outer model of V satisfying ZFC^-_{Ref} of the form $L[x]$ for a real x, in which every set is countable and the OIMH^-_{Ref} holds.

- We'll show that in $L[x]$ there is a real y coding a Δ_1-definable club of Σ_1-indiscernibles for L (when we say that a real y codes a Δ_1-definable club C we mean that C is Δ_1-definable with parameter y over $L[y]$).

- Then it follows from the OIMH^-_{Ref} in V that there is such a real in V, completing the proof.

- We begin our journey in $L[x]$, with the following:

Lemma 24. *Work in $L[x]$. Suppose that ϕ is a parameter-free formula with one free variable. Then for some Δ_1-definable (with real parameter) club C, either $\phi(\alpha)$ holds in L for all α in C or $\phi(\alpha)$ fails in L for all α in C.*

Proof Sketch of Lemma 24.

- Without loss of generality suppose that the class X of α such that $\phi(\alpha)$ holds in L is definably-stationary in $L[x]$ (i.e.X hits every $L[x]$-definable club).

- (Note that either X or its complement must be definably-stationary in $L[x]$, as otherwise we would obtain a contradiction from the existence of two disjoint clubs definable in $L[x]$.)

- Then over $L[x]$ we can force a club C through X such that $(L[x], C)$ satisfies ZFC^-_{Ref}: Conditions in $\mathbb{P}$ are closed subsets of X, ordered by end-extension.

- The forcing is ω-distributive, i.e. if $\langle D_i | i < \omega\rangle$ is a definable sequence of open dense classes, any condition p can be extended to a condition q belonging to each D_i.

- This is because by Reflection in $L[x]$ there is a definable club of ordinals C' such that for every $\alpha \in C'$, $\langle D_i \cap L_\alpha[x] | i \in \omega\rangle$ is dense in $\mathbb{P} \cap L_\alpha[x]$.

- By the definable stationarity of X we can choose such an α in X; then extend p ω-many times to conditions in $L_\alpha[x]$, hitting the various D_i.

- The union p_ω of these conditions together with α on top is a condition since α belongs to X and because taking the union of end-extending closed sets yields a closed set provided you add the relevant supremum (namely the supremum of the union).

- But as every set is countable (by the OIMH^-_{Ref} in $L[x]$) this shows that $\mathbb{P}$ is $(< Ord)$-distributive.

- This distributivity yields pretameness and therefore ZFC^-_{Ref} preservation (and indeed ZFC^-_{Ref} preservation relative to the generic club added).

- We can now further force over $(L[x], C)$ to add a real y so that C is Δ_1-definable over $L[y]$ with parameter y.

- This can be done with almost disjoint coding.

- To do the coding we need a definable class X such that each ordinal α is not only countable, but countable in $L[X \cap \alpha]$.

- But in the present setting, this is trivial as we can take the class X to simply be the real x.

- Moreover, in a general setting, to code a class X by a real with almost disjoint coding (when every set is countable) we need a sequence of distinct reals $\langle r_\alpha | \alpha \in Ord\rangle$ where each r_α can be defined just from the data given by $X \cap \alpha$.

- So if we have "decoded" $X \cap \alpha$ we can find r_α and then "decode" $X \cap (\alpha + 1)$.

- Then one can inductively "decode" all of X.

- In the present setting we can assume that C consists only of infinite ordinals and take X to be $x \cup C$ and take r_α to be the α-th real in the canonical well-order of $L[x]$.

- Then for all (infinite) α, $X \cap \alpha$ gives us x and therefore r_α.

- To code X by a generic real y, we replace each r_α by the set of codes for its finite initial segments (so that the various r_α are pairwise almost disjoint) and force the existence of a y with the property that α belongs to X iff y is almost disjoint from r_α.

- We now have an extension $L[x, y]$ in which C is Δ_1-definable from x and y. If desired, x and y can be combined into a single real z, with C Δ_1-definable in the parameter z over $L[z]$.

- All that needs to be checked (before we can pull back the inner model from $L[z]$ to $L[x]$ using the OIMH^-_{Ref} in $L[x]$) is that the forcing to add y over $L[x]$ preserves ZFC^-_{Ref}.

- But this follows from the fact that the almost disjoint coding has the Ord-chain condition, proving Lemma 24. $\square$

- We can now use Lemma 24 to show the existence of the indiscernibles required for Theorem 23.

- Using Lemma 24, for each Σ_1-formula ϕ with one free variable choose a Δ_1-definable (in some real parameter) club $C(\phi)$ so that either $\phi(\alpha)$ holds in L for all α in $C(\phi)$ or $\phi(\alpha)$ fails in L for all α in $C(\phi)$.

- Note that these choices can be made definably so the intersection C_1 of the various $C(\phi)$ is a definable club of ordinals with $\phi^L(\alpha)$ iff $\phi^L(\beta)$ for all $\alpha, \beta \in C_1$ and all Σ_1-formulas ϕ with one free variable.

- To describe these classes of indiscernibles the following definition will be useful:

Definition 25. A class X of ordinals is Σ_m-n-indiscernible for L if for any two increasing n-tuples $\vec{\alpha}$, $\vec{\beta}$ from X and any Σ_m-formula ϕ with n-many free variables:

$$\phi^L(\vec{\alpha}) \Leftrightarrow \phi^L(\vec{\beta})$$

- Using this terminology, we can describe C_1 as a club of Σ_1-1-indiscernibles for L.

- Again, using the methods of Theorem 20 and 22, we can force to make C_1 Δ_1-definable in a real and by the OIMH^-_{Ref} we have a Δ_1-definable club of Σ_1-1-indiscernibles for L in $L[x]$ (we'll also denote this club by 'C_1' for the sake of convenience).

- Now we want to go to more free variables before we intersect the clubs together to get the full L-indiscernibles required for $0^\sharp$.

- For each Σ_1-formula ϕ with parameter α in *two* free variables use Lemma 24 choose a Δ_1-definable (in a real parameter) club $C_1(\alpha, \phi)$ so that either $\phi(\alpha, \beta)$ holds in L for all β in $C_1(\alpha, \phi)$ or $\phi(\alpha, \beta)$ fails in L for all β in $C_1(\alpha, \phi)$.

- In the former case we say that α is ϕ-positive and in the latter case ϕ-negative. Either the first case holds for stationary-many γ or the second case holds for stationary-many γ (or both).

- By shooting a club we can ensure that either the first case holds for a club or the second case holds for a club (in either case, let the relevant club be C_2').

- We thin this club C_2' further by intersecting with the diagonal intersection of the various $C_1(\alpha, \phi)$ i.e. we take all β in C_2' which belong to $C_1(\alpha, \phi)$ for all $\alpha < \beta$ and all Σ_1-formulas ϕ with parameter α.

- Call this club C_2. Now if $\alpha < \beta$ and $\alpha^* < \beta^*$ are in C_2 and ϕ is a Σ_1-formula with two free variables we have:

$$\phi^L(\alpha, \beta) \Leftrightarrow \phi^L(\alpha, \beta^*)$$

- This holds because both β and β^* belong to $C_1(\alpha, \phi)$ iff $\phi^L(\alpha^*, \beta^*)$ (which in turn holds because either both α and α^* are ϕ-positive or both α and α^* are ϕ-negative). Thus, C_2 is a class of Σ_1-2-indiscernibles.

- Again applying the OIMH^-_{Ref} we can assume that C_2 is Δ_1-definable (in a real) in $L[x]$.

- We then repeat this to get Δ_1-definable clubs of Σ_1-3-indiscernibles by choosing $C_2(\alpha, \phi)$ to be a Δ_1-definable club such that either $\phi^L(\alpha, \beta, \gamma)$ holds for all $\beta < \gamma$ in $C_2(\alpha, \phi)$ or $\phi^L(\alpha, \beta, \gamma)$ fails for all $\beta < \gamma \in C_2(\alpha, \phi)$ and proceed as in the previous step to get a club C_3 which is Δ_1-definable in a real, consisting of Σ_1-3-indiscernibles for L.

- By repeating this procedure we get Δ_1-definable clubs of Σ_1-4-indiscernibles, Σ_1-5-indiscernibles and so on in $L[x]$.

- We continue this for ω-many steps and produce a definable sequence $\langle C_n | n < \omega \rangle$ of clubs which are Δ_1-definable in a real so that C_n consists of Σ_1-n-indiscernibles for L. Then the intersection $\bigcap_{n \in \omega} C_n$ is a definable club of Σ_1-indiscernibles for L.

Then we note:

Lemma 26. *(ZFC^-_{Ref}) Suppose that C is a club of Σ_1-indiscernibles for L (i.e. for a Σ_1-formula ϕ, $\phi(\vec{x})^L$ iff $\phi(\vec{y})^L$ for increasing tuples $\vec{x}$, $\vec{y}$ from C of the same length). Then C consists of Σ_ω- indiscernibles for L, i.e. for any ϕ, $\phi(\vec{x})^L$ iff $\phi(\vec{y})^L$ for increasing tuples $\vec{x}$,$\vec{y}$ from C of the same length.*

- Thus this club is also fully Σ_ω-indiscernible for L, and thus we have $0^\sharp$ in $L[x]$. □

Proof Sketch of Lemma 26.
- First we show that if α belongs to C then L_α is Σ_n-elementary in L for each n.

- Because C is a club and the class of α such that L_α is Σ_n-elementary in L is also a (definable) club, there are unboundedly many α in C such that L_α is Σ_n-elementary in L.

- In particular there are $\alpha < \beta$ in C such that L_α is Σ_n-elementary in L_β.

- But then by Σ_1-indiscernibility, L_α is Σ_n-elementary in L_β for all $\alpha < \beta$ in C, since "L_α is Σ_n-elementary in L_β" is a Σ_1^L-statement about the pair $\langle \alpha, \beta \rangle$.

- It follows that for each α in C, L_α is Σ_n-elementary in L because L is the limit of the Σ_n-elementary chain of L_α for α in C.

- Now suppose that ϕ is arbitrary and $\vec{x}$, $\vec{y}$ are tuples in C of the same length.

- Choose α in C greater than $\vec{x}, \vec{y}$. Now $\phi(\vec{x})^L$ is equivalent to $\phi(\vec{x})^{L_\alpha}$ because L_α is Σ_n-elementary in L.

- Moreover $\phi(\vec{x})^{L_\alpha}$ is equivalent to $\phi(\vec{y})^{L_\alpha}$ because $\langle \vec{x}, \alpha \rangle$ and $\langle \vec{y}, \alpha \rangle$ are increasing tuples from C of the same length and "$\phi(\vec{x})^{L_\alpha}$" is a Σ_1^L-statement about $\langle \vec{x}, \alpha \rangle$, and the same goes for "$\phi(\vec{y})^{L_\alpha}$".

- Finally, $\phi(\vec{y})^{L_\alpha}$ is equivalent to $\phi(\vec{y})^L$ because L_α is Σ_n-elementary in L.

- In conclusion, $\phi(\vec{x})^L$ iff $\phi(\vec{y})^L$, showing that C consists of Σ_ω-indiscernibles for L. $\square$

To sum up, we are now in a position where:

(1.) NBG^-_{Ref} + OIMH^-_{Ref} is provably consistent from ZFC + PD (Theorem 22), and

(2.) NBG^-_{Ref} + OIMH^-_{Ref} proves "$0^\sharp$ exists" (Theorem 23) and thus that ZFC with many large cardinal axioms added holds in L.

But how does this affect the prospects for countabilist foundations regarding the desiderata we identified earlier?

7 Set theory as a foundation under countabilism

- OK, so both the countabilist and uncountabilist look at each other and think the other misses out sets.

- The uncountabilist looks at the countabilist and thinks that they live in some tiny countable transitive model (or maybe $H(\omega_1)$, at best).

- The countabilist thinks the uncountabilist is blind to all sorts of collapses.

- How does set theory look as a *foundation* from this perspective?

- Let us start with **Generous Arena**.

- We can code the natural numbers exactly as before.

- The reals are now a proper class (so maybe you want to use NBG^- and its extensions).

- Although, one can use the fact that the algebraic reals (of which there are only countably many) are elementary in the reals.

- We can study the small countable object in learning about the large uncountable entity.

- (Contrast the case with the ZFC-ist's use of countable transitive models in studying V, cf. [Barton, 2020]).

- What then about third-order objects?

- Here we have a difference.

- Some third-order objects can be coded by second-order ones.

- e.g. The continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

- Other third-order (and higher-order) objects can be thought of as *only* living in (impoverished) ZFC models.

- The ZFC-theorist has exactly the same issue with the function space on all ordinals.

- So, whilst the interpretation is very *different*, it's not like the countabilist *can't* interpret the objects of ordinary mathematics.

- This then closely links in with **Shared Standard**—we can have a shared standard for interpreting proofs in set theory, it is just that the role of ZFC-based proofs is somewhat different and is to be interpreted as proving theorems about sets in some impoverished universe.

- **Metamathematical Corral** is also unaffected—we can study models of set theory (including ZFC models) exactly as before, either in the wider world of $\mathsf{NBG}^-_{Ref}/\mathsf{ZFC}^-_{Ref}$ or within some other ambient model of ZFC.

- **Risk Assessment** is somewhat more complicated.

- Whilst we have answered the **Motivational Challenge** via the idea of the universe being saturated under 'possible' sets (in line with **Maximality**) it should be noted that ZFC-based set theory has something more—a clear intuitive description of an underlying structure given by the iterative conception.

- This is partly (along with the long and unsuccessful attempt to find a contradiction) what convinces many that ZFC and its extensions is consistent.

- There are a few points to note here, and things are subtle.

- The first is that, strictly speaking, the countabilist can simply piggyback off the **Risk Assessment** provided by the iterative conception.

- This conception, she can contend, should indeed convince us that ZFC embodies a consistent conception of set.

- It is just that **Maximality** (for her) tells against the *truth* of ZFC.

- This response, whilst coherent, is somewhat unsatisfying.

- One might rightly complain that ZFC-based set theory still has an clear underlying conception where the countabilist perspective does not.

- This, one might think, speaks in favour of ZFC on independent grounds.

-Two possible responses:

(1.) Come up with some *non-iterative* conception underlying countabilism and ZFC^{-} and its extensions.

(2.) Modify or reinterpret the *iterative* conception to make it work on a countabilist perspective.

- There are various modal ways of providing a sort of iterative story for the countabilist (e.g. [Scambler, 2021]) one could piggy-back off.

- A second alternative is to reinterpret the notion of what it means to say *all possible sets*.

- What is required for the iterative conception is that when we grab "all possible subsets" at a successor stage, "all possible" does not coincide with "absolutely all".

- Rather we have to be grabbing at most countably-many at successor stages.

- In order for all sets to be included in this iterative process, we need to have the universe well-ordered in order type On.

- In that case we could define (letting R denote this well-order) the countabilist hierarchy V^C as follows:

Definition 27. The *Countabilist Iterative Hierarchy* is defined as follows:

$V_0^C = \emptyset$

$V_{\alpha+1}^C = Def(T_\alpha) \cup \{x\}$, where x is the R-least set not in V_α^C.

$V_\lambda^C = \bigcup_{\beta<\lambda} V_\beta^C$

$V^C = \bigcup_{\alpha \in On} V_\alpha^C$

- This would provide us with a hierarchy that stratifies the countabilist's universe in much the same way as the V_α hierarchy does for the uncountabilist.

- Unfortunately, this hierarchy is somewhat unsatisfying as is.

- First, we have no idea what R is like and indeed it is an additional commitment (one that seems to come from nowhere) beyond the normal prerequisites for an iterative hierarchy.

- Whilst the ZFC-ist needs the ordinals to generate the relevant V_α, this presentation of a countabilist hierarchy needs the ordinals *and* R.

- Second, the existence of such an R in the countabilist context is equivalent to CH.

- This, one might think, is an excessively specific assumption that needs to be used to generate a reasonable hierarchy.

- More promising might be the following idea: We iteratively add in collapses of the various sets to ω.

- The main problem here is to select the right collapse, but if one can stomach a Boolean universe one could build up a canonical universe

$V^{\mathbb{B}}$ consisting of Boolean-valued names so that for every name for a set there is also a name for a collapse of that set to ω.

- This leads to the idea that the universe looks like $V^{Col(\omega,<Ord)}$ where V is a inner model of ZFC (without names) and then the hierarchy is $V^{Col(\omega,<Ord)} = \bigcup_{\alpha \in Ord} V_\alpha^{Col(\omega,<\alpha)}$.

8 Conclusions and Open Questions

- We have argued that there are perspectives on **Maximality** in set theory on which every set is countable and on which set theory can perform many, if not all, of its usual foundational roles.

- This yields a very different picture of the nature of sets and how we interpret mathematics, but it doesn't seem clearly 'worse' than ZFC-based foundations.

- Some questions (first concerning interpretation:

Question. Are there *non-iterative* underlying conceptions that validate the countabilist perspective?

Further, if an iterative strategy is in fact desired:

Question. What are the available options for stratifying the hereditarily countable sets in order to provide an iterative picture for the countabilist, and how might they be philosophically motivated?

The next question concerns the mathematics of how the principles we have examined might be developed:

Question. Is it possible to find natural principles (e.g. by modifying the relevant absoluteness principle in question) that increase the large cardinal strength further (other than the brute force strategy)? Can this be done so as to yield more inner models with stronger large cardinals?

Finally, our results raise a (slightly more nebulous) general question:

Question. What do we really want set theory for? Is it merely a tool designed to fulfil certain foundational goals? Or is it part and parcel of our conception of set that it provide a study of *many* uncountably infinite cardinals?[1]

[1] [Friedman, 2016], for example, argues that there are three main roles for set theory: (1.) it is a branch of mathematics, (2.) it is a foundation for mathematics, and (3.) it provides a study of the set-concept. If we accept that each provides evidence for set-theoretic truth, progress could be made on the dialectic between the countabilist and uncountabilist.

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