Indeterminateness and 'The' Universe of Sets

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14. January 2021

- Over the last 150 years, we've seen that all known mathematics can be encoded within ZFC-based set theory (and other theories too!)
- This is suprising: a theory of how objects can be bolted together suffices to give us (sometimes ugly) avatars for any mathematical object desired.

This has borne the following fruits:

1. We have a single arena in which we can learn about the mathematical objects by relating them to one another.
2. If needed, we can use this arena to give us a shared standard for dispute resolution.
3. We can ascertain the consistency strength of new pieces of mathematics.

GREAT! SO TALK OF 'THE' UNIVERSE OF SETS SEEMS USEFUL. BUT IS IT REALLY SO UNPROBLEMATIC?

MAIN AIMS:

1. TO CONVINCE YOU THAT EXAMINING THIS QUESTION IS REALLY INTERESTING.
2. TO PROPOSE A (TENTATIVE) NEW DIRECTION.

STRUCTURE

§1 Problems of 'Height'.

§2 Problems of 'Width'.

§3 Multiversism & Potentialism

§4 Set-Theoretic Perspectivalism

§1 Problems of Height.

- We've known since Russell that not every condition determines a set (at least insofar as we accept classical logic)
- What became the 'standard' solution was the iterative conception of set.
- Philosophically: sets are formed in stages
- Mathematically: in ZFC we have $\forall x \exists \alpha\, x \in V_\alpha$

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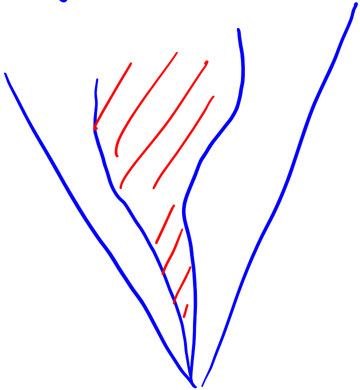
- There's still a question of what these 'collections' with members unboundedly in the V_α are.



ESPECIALLY WHEN:

(1.) WE CAN SAY APPARENTLY TRUE THINGS ABOUT THEM. e.g. $\{x \mid x \notin x\} = \{x \mid x = x\}$

(2.) WE CAN USE THEM TO DEFINE SETS. e.g. κ IS MEASURABLE IFF κ IS THE CRITICAL POINT OF A NON-TRIVIAL e.e. $j : V \longrightarrow M$.

(3.) WE CAN PROVE THEOREMS ABOUT THEM e.g. $\neg \exists j : V \longrightarrow V$

- THERE ARE LOTS OF INTERPRETATIONS OF CLASS-THEORETIC TALK THAT AVOID TREATING PROPER CLASSES AS "REAL" (e.g. IT'S JUST ABOUT FORMULAS HOLDING)
- LET'S ADOPT THE FOLLOWING:

METHODOLOGICAL MAXIM:
WHEREVER POSSIBLE, ALLOW FOR INTERPRETATIONS THAT ARE TRANSPARENT/HOMOPHONIC/NATURAL

- SIGH!: IF ONLY WE COULD TAKE $\mathcal{P}(V)$!

§2 Problems of Width

- In 1900, Number one on Hilbert's list of problems was to solve the Continuum Hypothesis (CH).
- In 1963 it was finally "resolved" after discoveries by Gödel (1940) that if ZFC is consistent then ZFC $\nvdash \neg$CH and Cohen that ZFC $\nvdash$ CH (assuming consistent)

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- Critical for the latter was the use of forcing.

- For this technique we:
  (1.) Take a model $\mathcal{M}$ of ZFC.
  (2.) Use a partial order $\mathbb{P} \in \mathcal{M}$ to provide names for potential new objects.
  (3.) Explain how names can be evaluated using a generic filter $G \subseteq \mathbb{P}$, $G \notin \mathcal{M}$.

- This gives us a flexible yet controlled way of changing truth.

FACT Let $\mathcal{M} \models ZFC$. Then $\mathcal{M}$ has forcing extensions $\mathcal{M}[G]$ and $\mathcal{M}[H]$ such that:

- $\mathcal{M}[G] \models \neg CH$, collapsing no cardinals but adding reals.
- $\mathcal{M}[H] \models CH$, adding no new reals but collapsing cardinals.

- But isn't all this just to do with axioms & proofs? ZFC $\nvdash$ CH, ZFC $\nvdash \neg$CH etc.

- Again, we can:
  (1.) Say apparently true things, e.g. $G \notin V$ for generic G.

  (2.) Formulate axioms: e.g. A cardinal κ is virtually extendible iff $\forall \alpha > \kappa$ in a set-forcing extension there is a e.e. $j: V_\alpha \rightarrow V_\beta$, $crit(j) = \kappa$, $j(\kappa) > \alpha$.

  (3.) Prove theorems: e.g [Malliaris & Shelah, 2015] $\mathfrak{p} = \mathfrak{t}$.

- Sigh: If only we could add subsets to V!

§3 MULTIVERSISM & POTENTIALISM.

- ONE KIND OF RESPONSE: MULTIVERSIM

"THE MULTIVERSE VIEW IS ONE OF HIGHER-ORDER REALISM — PLATONISM ABOUT UNIVERSES — AND I DEFEND IT AS A REALIST POSITION ASSERTING ACTUAL EXISTENCE OF UNIVERSES"

— JOEL DAVID HAMKINS 'THE SET-THEORETIC MULTIVERSE' (2012)

- We then have (among others):
  - Height Multiversism: If V is a universe then there is another universe V' such that $V \in V'$.
  - Forcing Multiversism: If V is a universe and $\mathbb{P}$ a forcing p.o. in V, then there is a $V' = V[G]$ s.t. $G \in V' = V[G]$ and G is V-generic.

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- THESE VIEWS MOTIVATE MATHEMATICS.
  - HEIGHT MULTIVERSISM: TAIT-STYLE REFLECTION
  - FORCING MULTIVERSISM: MULTIVERSE AXIOMATISATIONS.

e.g. STEEL'S MV-AXIOMS: VARIABLES FOR SETS $(x_0, \ldots)$ AND UNIVERSES $(V, W, V_0, \ldots)$

(i) φ^W FOR EVERY AXIOM φ OF ZFC

(ii)(a) EVERY UNIVERSE IS A TRANSITIVE PROPER CLASS.

(b) IF $\mathbb{P} \in W$ THEN THERE IS A UNIVERSE $W[G]$.

(c) IF W IS A UNIVERSE AND $W = U[G]$ THEN U IS A UNIVERSE.

(d) IF W AND V ARE UNIVERSES, THEN THERE ARE G, H GENERIC FOR $\mathbb{P}_W \in W$ AND $\mathbb{P}_V \in V$ S.T. $W[G] = V[H]$

- MULTIVERSISM FACES THE FOLLOWING PROBLEM: WHY CAN'T I PUT THE UNIVERSES TOGETHER?
- MAYBE BECAUSE I CAN'T QUANTIFY OVER ALL SETS? UH OH!
- A RESPONSE: POTENTIALISM.

"SET-THEORETIC POTENTIALISM IS THE VIEW IN THE PHILOSOPHY OF MATHEMATICS THAT THE UNIVERSE OF SET THEORY IS NEVER FULLY COMPLETED, BUT RATHER UNFOLDS GRADUALLY..." - HAMKINS & LINNEBO 'THE MODAL LOGIC OF SET-THEORETIC POTENTIALISM AND THE POTENTIALIST MAXIMALITY PRINCIPLES'

- KEY IS THE USE OF MODAL RESOURCES e.g. $\Diamond$, $\Box$, AND THINKING OF THE SUBJECT-MATTER OF SET THEORY AS SOMETHING LIKE A KRIPKE FRAME.

HEIGHT POTENTIALISM:

- WORLDS = $\{V_\beta \mid \beta$ IS AN ORDINAL$\}$
- $\Diamond\varphi$ = TRUE IN SOME LARGER V_β

FORCING POTENTIALISM

- WORLDS = $\{V[G] \mid G$ IS V-GENERIC$\}$
- $\Diamond\varphi$ = TRUE IN SOME FORCING EXTENSION.

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- We can prove theorems about these systems.
  - Theorem (Hamkins, Löwe) The modal logic of forcing is S4.2.
  - Theorem (Hamkins, Linnebo) The modal logic of height (rank) potentialism is S4.3.
- And formulate axioms: e.g

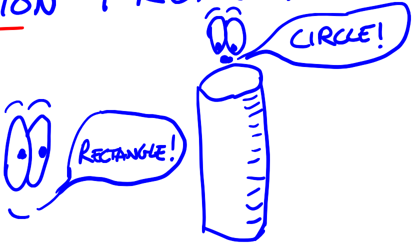
  Maximality Principle: $\Diamond\Box\varphi \to \varphi$

(with real parameters, equiconsistent with "Ord is Mahlo").

§4 Set-Theoretic Perspectivalism.

- We've seen that each view motivates interesting mathematics.
- This includes the universe/actualist view: look at attempts to propose new axioms for CH!
- But by our methodological assumption this means we should (if possible) take the entities of each seriously.
- But they seem incompatible

- A proposed direction of research: Set-theoretic perspectivalism: There are multiple equally legitimate views on the nature of sets. Each view is a reasonable expression from a particular viewpoint.



- MAYBE WE SHOULD THINK ABOUT THE CONTENT OF SET-THEORETIC ASSERTIONS NOT AS A MATTER OF JUST USING "$\in$", BUT RATHER AS A MINGLING OF CONCEPT & LANGUAGE
- NOTE: EACH VIEW CAN INTERPRET THE OTHER, SO WE STILL HAVE ARENAS FOR MATHEMATICS.

Conclusion

- Set theory and philosophy can work together to enhance & enrich each other.
- There's the possibility of viewing these debates on ontology as perspectival.
- But what of our shared standard?

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Thanks for listening!
I am very grateful to:
VolkswagenStiftung, FWF, AHRC, Carolin Antos, Andrés Caicedo, Monroe Eskew, Sy-David Friedman, Deborah Kant, Daniel Kuby, Sam Roberts, Jonathan Schilhan, Robin Solberg, Kameryn Williams