Intensional Classes and Intuitionistic Topoi

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- Second: This talk contains a lot of diagrams, and I'm rubbish at using TikZ, so...

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APOLOGIES FOR THE HANDWRITING!

I SECRETLY LIKE DOING SLIDES THIS WAY

Intensional Classes and Intuitionistic Topoi

- FOR A LONG TIME WE'VE KNOWN ABOUT (AND MAYBE BEEN TROUBLED BY) THE SET-THEORETIC PARADOXES.
- THERE'S LOTS OF CONDITIONS (e.g. $x \notin x$, $x = x$, "x IS AN ORDINAL") THAT DO NOT APPEAR TO DETERMINE SETS.
- BUT WE APPEAR TO BE ABLE TO SAY TRUE THINGS ABOUT THEM: e.g. $\{x \mid x = x\} = \{x \mid x \notin x\}$
- AND WE APPEAR TO USE THEM e.g. 1. $\exists j : V \to M$, $crit(j) = \kappa$

  e.g. 2. $\neg \exists j : V \to V$.

Intensional Classes and Intuitionistic Topoi

- On pain of contradiction they are not sets.
- So what makes them different?

The intensional conception of classes

(A) Sets are extensional entities: their identity is determined by membership (which cannot change)

(B) Classes are intensional entities: their identity is given by application conditions, which can change membership.

Intensional Classes and Intuitionistic Topoi

MAIN QUESTIONS:

(1) HOW CAN WE FORMALISE THIS TALK OF INTENSIONAL CLASSES?

(2) CAN SUCH A FORMALISM SHED LIGHT ON WHAT LOGIC SHOULD GOVERN THEM (CONTINGENT ON OTHER VIEWS)?

§1 POTENTIALISM & INTENSIONAL CLASSES

§2 TOPOS THEORY

§3 LAWVERE'S VARIABLE SETS

§4 APPLICATIONS TO POTENTIALISM

§5 MY LETTER TO SANTA (OPEN Qs)

MAIN CLAIM

THE FORMALISATION OF (SOME) CLASSES IN TERMS OF FUNCTOR CATEGORIES SUGGESTS THAT THE LOGIC DEPENDS ON THE POTENTIALISM

§1 Potentialism & Intensional Classes.

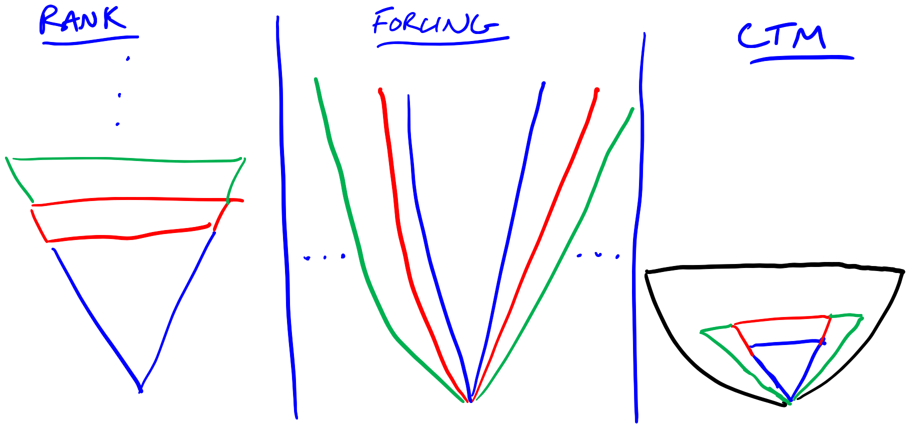
Potentialism is the view that 'the' universe of sets is never completed, but rather unfolds gradually as parts of it come into existence or become accessible (Adapted from Hamkins & Linnebo)

- Forcing potentialism is the view that we can always 'see' more of the universe by adding (set) generics.
- Rank potentialism is the view that we can always 'see' more of the universe by making the current world a V_κ of something taller

- THERE'S MANY MORE BESIDES (TRANSITIVE $\mathcal{M} \models ZFC$, COUNTABLE TRANSITIVE $\mathcal{M} \models ZFC$, COUNTABLE $\mathcal{M} \models ZFC$)
- THESE ROUGH PHILOSOPHICAL IDEAS MOTIVATE THE CONSIDERATION OF FORMAL POTENTIALIST SYSTEMS CONCEIVED OF AS KRIPKE FRAMES.

RANK POTENTIALISM	FORCING POTENTIALISM	CTM POTENTIALISM
$\mathcal{V} = \{V_\beta \mid \beta \in On\}$	$\mathcal{V} = \{V[G] \mid G$ IS V-GENERIC OR $V[G]=V\}$	$\mathcal{V} = \{\mathcal{M} \mid$ "$\mathcal{M} \models ZFC$ IS A CTM"$\}$
$\Diamond\varphi =$ "φ TRUE IN LARGER V_β"	$\Diamond\varphi =$ "φ TRUE IN A FORCING EXTENSION"	$\Diamond\varphi =$ "TRUE IN LARGER CTM"
$w_1 R w_2$ IFF $\alpha \leq \beta$, $w_1 = V_\alpha$, $w_2 = V_\beta$	$w_1 R w_2$ IFF w_2 IS A FORCING EXTENSION OF w_1	$w_1 R w_2$ IFF $\mathcal{M}_1 \subseteq \mathcal{M}_2$, $w_1 = \mathcal{M}_1$, $w_2 = \mathcal{M}_2$

- SO HOW CAN WE HAVE INTENSIONAL CLASSES?
  ↳ THESE VARY THEIR EXTENSION ACROSS WORLDS.



§2 Topos Theory

- Topos theory is a branch of category theory that can be thought of in (at least) two ways:
  1. It's a way of bringing insights from (algebraic) geometry to bear on logic.
  2. It's a way of isolating the core logical properties instantiated by the sets, and then seeing how these behave in other contexts

- A CATEGORY COMPRISES:

1. A COLLECTION OF OBJECTS
2. A COLLECTION OF ARROWS BETWEEN OBJECTS THAT:
   - 2(i) IS CLOSED UNDER AN ASSOCIATIVE COMPOSITION OPERATION $\circ$
   - 2(ii) CONTAINS FOR EVERY OBJECT x AN IDENTITY MORPHISM Id_x s.t. $\forall f : x \to y$, $\forall g\, z \to x$, $f \circ Id_x = f$, $Id_x \circ g = g$

- USING THE LANGUAGE OF CATEGORIES WE CAN ANALYSE THE KINDS OF TRANSFORMATION AVAILABLE IN MANY DIFFERENT CONTEXTS.
- IN PARTICULAR, WE CAN ISOLATE SIMILARITIES.
- THIS IS KEY FOR ANALYSING HOW SOME OF THE LOGICAL PROPERTIES OF SET (THE CATEGORY OF ALL SETS) TRANSFER TO A BROADER CONTEXT.

An (elementary) topos is a category $\mathcal{E}$ s.t.

1. $\mathcal{E}$ has finite limits.
2. $\mathcal{E}$ is cartesian closed
3. $\mathcal{E}$ has a subobject classifier.

Intensional Classes and Intuitionistic Topoi

A SUBOBJECT CLASSIFIER IS THE KEY SYSTEM OF MAPPINGS THAT ALLOWS US (IN A TOPOS) TO TALK CATEGORIALLY ABOUT TRUTH VALUES.

- A MONIC $f : a \rightarrowtail d$ IS A $\mathcal{C}$-ARROW S.T. $\forall g, h : b \to a,\ f \circ g = f \circ h \Rightarrow g = h$ (IN SET, THINK INJECTIONS AND $\subseteq$)

- A TERMINAL OBJECT IS AN OBJECT 1 S.T. FOR ANY $\mathcal{C}$-OBJECT a THERE IS A UNIQUE $\mathcal{E}$-ARROW $! : a \to 1$ (IN SET, THINK SINGLETONS)

FOR $f:a \to c$ AND $g:b \to c$ THE <u>PULLBACK OF f,g</u> IS A PAIR OF ARROWS $g':d \to a$ AND $f':d \to b$ S.T.

(i) $f \circ g' = g \circ f'$

(ii) WHENEVER $h:e \to a$ AND $i:e \to b$ ARE S.T. $f \circ h = g \circ j$, THEN THERE IS A <u>UNIQUE</u> $k:e \to d$

$$\begin{tikzcd}
e \arrow[r, dashed, "k"] \arrow[rr, bend left, "j"] \arrow[dr, bend right, "h"'] & d \arrow[r, "f'"] \arrow[d, "g'"'] & b \arrow[d, "g"] \\
 & a \arrow[r, "f"'] & c
\end{tikzcd}$$

(IN SET LET $d = \{\langle x,y\rangle \mid x \in a, y \in b$ AND $f(x) = g(y)\}$ AND f' AND g' BE THE PROJECTIONS $f'(\langle x,y\rangle) = y$, $g'(\langle x,y\rangle) = x$)

IF $\mathcal{C}$ IS A CATEGORY W/ A TERMINAL OBJECT 1 THEN A SUBOBJECT CLASSIFIER FOR $\mathcal{C}$ IS A $\mathcal{C}$-OBJECT Ω TOGETHER W/ A $\mathcal{C}$-ARROW $\mathrm{true}: 1 \to \Omega$ THAT SATISFIES THE:

$\boxed{\Omega\text{-AXIOM}}$ FOR EACH MONIC $f: a \rightarrowtail d$ THERE IS EXACTLY ONE $\mathcal{C}$-ARROW $\chi_f : d \to \Omega$ S.T.

$$\begin{tikzcd} a \arrow[r, tail, "f"] \arrow[d, "!"'] & d \arrow[d, "\chi_f"] \\ 1 \arrow[r, "\mathrm{true}"] & \Omega \end{tikzcd}$$

IS A PULLBACK SQUARE

IN SET, THINK OF f AS AN INCLUSION, Ω AS $\{0,1\}$, true AS PICKING $1 \in \{0,1\}$ AND χ_f AS A CHARACTERISTIC FUNC

- Topoi are many and varied (finite sets, sheaves, etc.).
- The core point is that we get (two) kinds of truth about subsets from the subobject classifier.
- In Set the collection of subobjects of some set D is just $\mathcal{P}(D)$ and this forms a Boolean algebra
- In a topos it forms a Heyting algebra

e.g. COMPLEMENTS

AN INITIAL OBJECT IS A $\mathcal{C}$-OBJECT 0 S.T. FOR ANY $\mathcal{C}$-OBJECT a THERE IS A UNIQUE $\mathcal{C}$-ARROW

$$!_0 : 0 \longrightarrow a$$

$$\bot = \chi_{!_0}$$

$\neg : \Omega \longrightarrow \Omega$ IS THE UNIQUE $\mathcal{C}$-ARROW S.T.

$$\begin{tikzcd} 1 \arrow[r, "\bot"] \arrow[d, "!"'] & \Omega \arrow[d, "\neg = \chi_\bot"] \\ 1 \arrow[r, "\top"] & \Omega \end{tikzcd}$$

WE DO SIMILAR MOVES WITH:

$\cap, \cup, \Rightarrow : \Omega \times \Omega \longrightarrow \Omega$

- WE CAN THEN USE THE IDEA OF THESE ARROWS TO PROVIDE VALUATIONS.
- AN $\mathcal{E}$-VALUATION ASSIGNS EACH SENTENCE AN ARROW $v : 1 \longrightarrow \Omega$ w/ COMPOSITIONAL RULES OBEYING ARROW-THEORETIC ONES IN $\mathcal{E}$.

e.g. $V(\neg\alpha) = \neg \circ V(\alpha)$

$$\begin{tikzcd} 1 \arrow[r, "V(\alpha)"] \arrow[dr, "V(\neg\alpha)"'] & \Omega \arrow[d, "\neg"] \\ & \Omega \end{tikzcd}$$

WE SAY $\mathcal{E} \models \alpha$ IFF ALL $\mathcal{E}$-VALS $V(\alpha) = T : 1 \longrightarrow \Omega$

- THEN, GIVEN A SUBOBJECT $f : a \rightarrowtail d$ THE COMPLEMENT OF f (RELATIVE TO d) IS $-f : -a \rightarrowtail d$ WHOSE CHARACTER IS $\neg \circ \chi_f$

$$\begin{tikzcd}
-a \arrow[r, tail, "-f"] \arrow[d] & d \arrow[d, "\neg \circ \chi_f"] \\
1 \arrow[r, "\top"'] & \Omega
\end{tikzcd}$$

GIVEN $f: a \rightarrowtail d$ AND $g: b \rightarrowtail d$ WE PUT $f \leq g$ IFF THERE IS A $\mathcal{E}$-ARROW $h: a \to b$ W/

$$\begin{tikzcd} b \arrow[dr, tail, "g"] & \\ & d \\ a \arrow[uu] \arrow[ur, tail, "f"'] & \end{tikzcd}$$

SETTING $\mathrm{SUB}(D) = \{ [f]_{\leq} \mid \exists a \; f: a \rightarrowtail d \}$

$$(\mathrm{SUB}(D), \cap, \cup, \Rightarrow)$$

IS A HEYTING ALGEBRA.

WE CAN THEN LINK SUBOBJECTS TO VALUATIONS

e.g. $\mathcal{E} \models \alpha \vee \neg \alpha$ IFF $\mathrm{SUB}(1)$ IS A BA.

§3 LAWVERE'S VARIABLE SETS

- WE'LL SEE THAT WE CAN VIEW INTENSIONAL CLASSES THROUGH THE LENS OF PARTICULAR TOPOI

A FUNCTOR BETWEEN CATEGORIES $\mathcal{C}$ AND $\mathcal{D}$ IS A MAP THAT ASSIGNS

(i) TO EACH $\mathcal{C}$-OBJECT c A $\mathcal{D}$-OBJECT $F(c)$.

(ii) TO EACH $\mathcal{C}$-ARROW $g: a \to b$ A $\mathcal{D}$-ARROW $F(g)$ S.T.

(a) $F(Id_a) = Id_{F(a)}$

(b) $F(g \circ f) = F(g) \circ F(f)$

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- A natural transformation between $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{C} \to \mathcal{D}$ is an assignment τ that provides, for each $\mathcal{C}$-object a, a $\mathcal{D}$-arrow $\tau_a : F(a) \longrightarrow G(a)$, such that for any $\mathcal{C}$-arrow $f: a \to b$

$\mathcal{C}$

$$\begin{tikzcd} a \arrow[d, "f"] \\ b \end{tikzcd}$$

$\mathcal{D}$

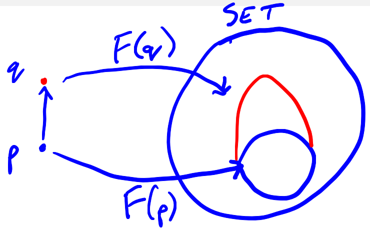
$$\begin{tikzcd} F(a) \arrow[r, "\tau_a"] \arrow[d, "F(f)"'] & G(a) \arrow[d, "G(f)"] \\ F(b) \arrow[r, "\tau_b"] & G(b) \end{tikzcd}$$

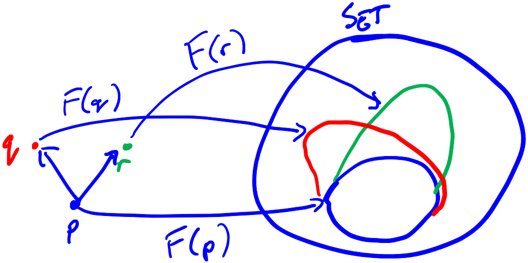
Intensional Classes and Intuitionistic Topoi

- THE FUNCTOR CATEGORY $\mathcal{D}^{\mathcal{C}}$ HAS
  AS (i) OBJECTS FUNCTORS $F: \mathcal{C} \longrightarrow \mathcal{D}$
  (ii) ARROWS NATURAL TRANSFORMATIONS
  $\tau: F \longrightarrow G$

- IN DEVELOPING THESE IDEAS LAWVERE DISCOVERED THAT:
  1. YOU CAN THINK OF A PARTIAL-ORDER $\mathbb{P}$ AS A CATEGORY w/ $f: p \to q$ IFF $p \leq_{\mathbb{P}} q$
  2. THE FUNCTOR CATEGORY $\mathrm{SET}^{\mathbb{P}}$ REPRESENTS VARIABLE SETS THROUGH "TIME."

Intensional Classes and Intuitionistic Topoi





FACT - $SET^{\mathcal{C}}$ IS A TOPOS FOR ANY SMALL $\mathcal{C}$

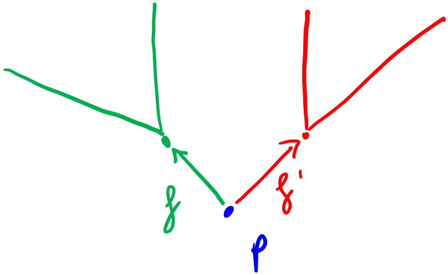
- IN THE CASE OF $SET^{\mathbb{P}}$ A MONIC $\tau : F \rightarrowtail G$ IS COMPOSED OF INJECTIVE COMPONENTS $\tau_p : F(p) \rightarrowtail G(p)$

- $\Omega(p)$ = THE SET OF p-SIEVES, WHERE A p-SIEVE IS A $S \subseteq \{ f \mid$ FOR SOME q, $f : p \to q$ IN $\mathbb{P} \}$ S.T. IF $f \in S$ THEN $g \circ f \in S$

- $\Omega_{pq} : \Omega_p \longrightarrow \Omega_q$ IS DEFINED COMPONENT WISE BY TAKING A p-SIEVE S TO $S \cap [q)$

INTENSIONAL CLASSES AND INTUITIONISTIC TOPOI

- TRUE: $1 \longrightarrow \Omega$ PICKS THE LARGEST p-SIEVE FOR ANY p.



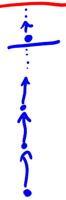
- WILL F BE A SUBOBJECT OF G?

Intensional Classes and Intuitionistic Topoi

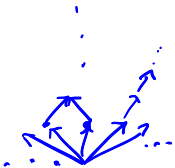
§4 Applications to Potentialism.

- We think of $\mathbb{P}$ as the partial order on (a set-sized piece of) the worlds in a potentialist system

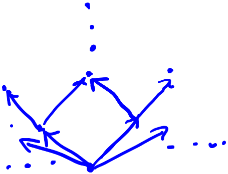
Rank



Forcing



CTM



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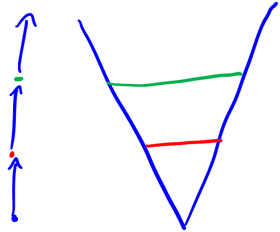
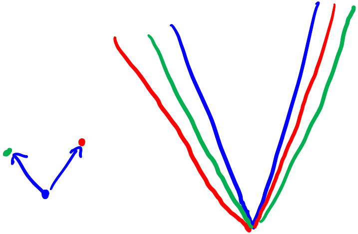
- AND THE POINTS IN $\mathbb{P}$ AS MOMENTS OF TIME IN THE POTENTIALIST SYSTEM





- KEY POINT: THE STRUCTURE OF $\mathbb{P}$ IS INTIMATELY RELATED TO THE STRUCTURE OF THE SUBOBJECT CLASSIFIER

FACT (DUMMETT, SEGERBERG) $\mathbb{P}$ IS INFINITE AND LINEARLY ORDERED IFF THE LOGIC OF $\mathcal{E}$ IS INTUITIONISTIC LOGIC WITH

$$(\varphi \to \psi) \vee (\psi \to \varphi)$$

ADDED.

THIS MEANS:

1. THE LOGICAL PROPERTIES OF INTENSIONAL CLASSES (IN A ROUGH SENSE) ARE INTUITIONISTIC.
2. BOTH THE FORCING POTENTIALIST AND CTM-POTENTIALIST LOSE:

$$(\varphi \longrightarrow \psi) \vee (\psi \longrightarrow \varphi)$$

§5 Neil's Letter To Santa.

Dear Santa, This year, I would like:

1. A full understanding of how the particular potentialist $\mathbb{P}$ relate to the properties of the S.O. classifier.
   - Forcing potentialism has incompatibility. CTM potentialism does not.
   - Both can have least elements.

2. A COMPARISON (AND BETTER UNDERSTANDING OF) THE INTERNAL AND EXTERNAL LOGICS.
3. A WAY OF HANDLING PROPER-CLASS-SIZED $\mathbb{P}$.
4. A WAY OF DEALING WITH CLASSES LOSING ELEMENTS.
5. AN ACCOUNT OF HOW THIS LINKS W/ FORCING ($\text{SET}^{\mathbb{P}^{op}}$).

CONCLUSIONS

- REPRESENTING INTENSIONAL CLASSES THIS WAY INDICATES THAT THEIR LOGIC IS NON-CLASSICAL.
- THERE'S A BUNCH OF (QUASI) OPEN QUESTIONS
- THERE IS A PHILOSOPHICAL CHALLENGE OF HOW THE POTENTIALIST SHOULD REACT.
  - 1. OBJECT TO CHARACTERISATION
  - 2. ACCEPT NON-CLASSICALITY.
  - 3. REFINE ACCOUNT OF INTENSIONAL CLASSES.

Thanks

Thanks for listening, I look forward to the comments!

Hugely grateful to:

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