

INTENSIONAL CLASSES AND INTUITIONISTIC TOPOI

Neil Barton
Universität Konstanz



VolkswagenStiftung

Universität
Konstanz



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INTENSIONAL CLASSES AND INTUITIONISTIC TOPOI

- THANK YOU FOR THE INVITATION!

SPOILER ALERT:

- THIS TALK CONTAINS A LOT OF DIAGRAMS.
- I AM RUBBISH WITH TIKZ.
- PLEASE EXCUSE THE HANDWRITING

SECRETLY I LIKE
DOING SLIDES THIS
WAY!

INTENSIONAL CLASSES AND INTUITIONISTIC TOPOI

- FOR A LONG TIME WE'VE KNOWN ABOUT (AND MAYBE BEEN TROUBLED BY) THE SET-THEORETIC PARADOXES.
- THERE'S LOTS OF CONDITIONS (e.g. $x \notin x$, $x = x$, " x IS AN ORDINAL") THAT DO NOT APPEAR TO DETERMINE SETS.
- BUT WE APPEAR TO BE ABLE TO SAY TRUE THINGS ABOUT THEM: e.g. $\{x | x = x\} = \{x | x \neq x\}$
- AND WE APPEAR TO USE THEM e.g. $\exists j: V \rightarrow \mathcal{P}(V)$, $\text{crit}(j) = K$
e.g. 2. $\neg \exists j: V \rightarrow V$.

INTENSIONAL CLASSES AND INTUITIONISTIC TOPOI

- On PAIN OF CONTRADICTION THEY ARE NOT SETS.
- So WHAT MAKES THEM DIFFERENT?

THE INTENSIONAL CONCEPTION OF CLASSES

- (A) SETS ARE EXTENSIONAL ENTITIES: THEIR IDENTITY IS DETERMINED BY MEMBERSHIP (WHICH CANNOT CHANGE)
- (B) CLASSES ARE INTENSIONAL ENTITIES: THEIR IDENTITY IS GIVEN BY APPLICATION CONDITIONS, WHICH CAN CHANGE MEMBERSHIP.

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MAIN QUESTIONS:

(1) How CAN WE FORMALISE THIS TALK OF INTENSIONAL CLASSES?

(2) CAN SUCH A FORMALISM SHED LIGHT ON WHAT LOGIC SHOULD GOVERN THEM (CONTINGENT ON OTHER VIEWS)?

§1 POTENTIALISM & INTENSIONAL CLASSES MAIN CLAIM

§2 TOPOS THEORY

§3 LAWVERE'S VARIABLE SETS

§4 APPLICATIONS TO POTENTIALISM

§5 MY LETTER TO SANTA (OPEN Qs)

THE FORMALISATION
OF (SOME) CLASSES IN TERMS
OF FUNCTOR CATEGORIES
SUGGESTS THAT THE LOGIC
DEPENDS ON THE POTENTIALISM

INTENSIONAL CLASSES AND INTUITIONISTIC TOPOI

§ 1 POTENTIALISM & INTENSIONAL CLASSES.

POTENTIALISM IS THE VIEW THAT 'THE' UNIVERSE OF SETS IS NEVER COMPLETED, BUT RATHER UNFOLDS GRADUALLY AS PARTS OF IT COME INTO EXISTENCE OR BECOME ACCESSIBLE (ADAPTED FROM HUMKING & LINNARD)

- FORCING POTENTIALISM IS THE VIEW THAT WE CAN ALWAYS 'SEE' MORE OF THE UNIVERSE BY ADDING (SET) GENERICS.
- RANK POTENTIALISM IS THE VIEW THAT WE CAN ALWAYS 'SEE' MORE OF THE UNIVERSE BY MAKING THE CURRENT WORLD A V_K OF SOMETHING TALLER

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- THERE'S MANY MORE BESIDES (TRANSITIVE $\mathcal{R} \models \text{ZFC}$, COUNTABLE TRANSITIVE $\mathcal{R} \models \text{ZFC}$, COUNTABLE $\mathcal{R} \models \text{ZFC}$)
- THESE ROUGH PHILOSOPHICAL IDEAS MOTIVATE THE CONSIDERATION OF FORMAL POTENTIALIST SYSTEMS CONCEIVED OF AS KRIIPKE FRAMES.

RANK POTENTIALISM

$$V = \{V_\beta \mid \beta \in \text{On}\}$$

$$\Diamond \varphi = \text{"}\varphi \text{ TRUE IN LARGER } V_\beta \text{"}$$

$$w_1 R w_2 \text{ IFF } \alpha \leq \beta$$

$$\begin{array}{c} \text{"} \\ \sqrt{\alpha} \end{array} \quad \begin{array}{c} \text{"} \\ \sqrt{\beta} \end{array}$$

FORCING POTENTIALISM

$$V = \{V[G] \mid G \text{ IS } V\text{-GENERIC} \text{ OR } V[G] = V\}$$

$$\Diamond \varphi = \text{"}\varphi \text{ TRUE IN A FORCING EXTENSION"}$$

$$w_1 R w_2 \text{ IFF } w_2 \text{ IS A FORCING EXTENSION OF } w_1$$

CTM POTENTIALISM

$$V = \{\mathcal{R} \mid \mathcal{R} \models \text{ZFC IS A CTM}\}$$

$$\Diamond \varphi = \text{"TRUE IN LARGER CTM"}$$

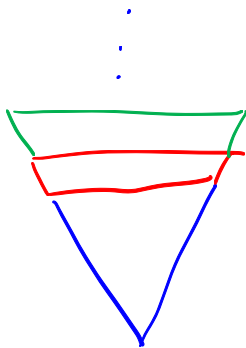
$$w_1 R w_2 \text{ IFF } \mathcal{R}_1 \leq \mathcal{R}_2$$

$$\begin{array}{c} \text{"} \\ \mathcal{R}_1 \end{array} \quad \begin{array}{c} \text{"} \\ \mathcal{R}_2 \end{array}$$

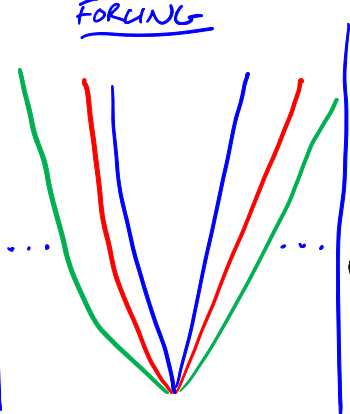
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- SO HOW CAN WE HAVE INTENSIONAL CLASSES?
↳ THESE VARY THEIR EXTENSION ACROSS WORLDS.

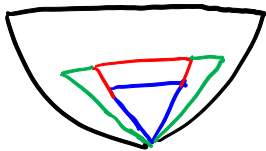
RANK



FORCING



CTM



§2 TOPOS THEORY

- TOPOS THEORY IS A BRANCH OF CATEGORY THEORY THAT CAN BE THOUGHT OF IN (AT LEAST) TWO WAYS:
 - 1. IT'S A WAY OF BRINGING INSIGHTS FROM (ALGEBRAIC) GEOMETRY TO BEAR ON LOGIC.
 - 2. IT'S A WAY OF ISOLATING THE CORE LOGICAL PROPERTIES INSTANTIATED BY THE SETS, AND THEN SEEING HOW THESE BEHAVE IN OTHER CONTEXTS

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- A CATEGORY COMPRISES:

1. A COLLECTION OF OBJECTS
2. A COLLECTION OF ARROWS BETWEEN OBJECTS
THAT:

└ 2(i) IS CLOSED UNDER AN ASSOCIATIVE COMPOSITION
OPERATION $\circ$

└ 2(ii) CONTAINS FOR EVERY OBJECT x
AN IDENTITY MORPHISM id_x s.t.

$$\forall f: x \rightarrow y, \forall g: z \rightarrow x, f \circ id_x = f, id_x \circ g = g$$

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- USING THE LANGUAGE OF CATEGORIES WE CAN ANALYSE THE KINDS OF TRANSFORMATION AVAILABLE IN MANY DIFFERENT CONTEXTS.
- IN PARTICULAR, WE CAN ISOLATE SIMILARITIES.
- THIS IS KEY FOR ANALYSING HOW SOME OF THE LOGICAL PROPERTIES OF SET (THE CATEGORY OF ALL SETS) TRANSFER TO A BROADER CONTEXT.

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AN (ELEMENTARY) TOPOS IS A CATEGORY $\mathcal{E}$ S.T.

- 1. $\mathcal{E}$ HAS FINITE LIMITS.
- 2. $\mathcal{E}$ IS CARTESIAN CLOSED
- 3. $\mathcal{E}$ HAS A SUBOBJECT CLASSIFIER.

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A SUBJECT CLASSIFIER IS THE KEY SYSTEM OF MAPPINGS THAT ALLOWS US (IN A TOPOS) TO TALK CATEGORICALLY ABOUT TRUTH VALUES.

- A MONIC $f: a \rightarrow d$ IS A $\mathcal{C}$ -ARROW S.T.
 $\forall g, h: b \rightarrow a, f \circ g = f \circ h \Rightarrow g = h$ (IN SET, THINK INJECTIONS AND $\subseteq$)

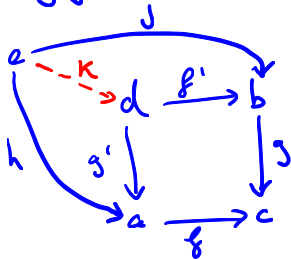
- A TERMINAL OBJECT IS AN OBJECT 1 S.T. FOR ANY $\mathcal{C}$ -OBJECT a THERE IS A UNIQUE $\mathcal{C}$ -ARROW $! : a \rightarrow 1$ (IN SET, THINK SINGLETONS)

INTENSIONAL CLASSES AND INTUITIONISTIC TOPOI

FOR $f:a \rightarrow c$ AND $g:b \rightarrow c$ THE PULLBACK OF f, g IS A PAIR OF ARROWS $g':d \rightarrow a$ AND $f':d \rightarrow b$ S.T.

(i) $f \circ g' = g \circ f'$

(ii) WHENEVER $h:e \rightarrow a$ AND $i:e \rightarrow b$ ARE S.T. $f \circ h = g \circ i$, THEN THERE IS A UNIQUE $k:e \rightarrow d$



IN SET LET
 $d = \{ \langle x, y \rangle \mid x \in a, y \in b \text{ AND } f(x) = g(y) \}$
AND f' AND g' BE THE
PROJECTIONS $f'(\langle x, y \rangle) = y$
 $g'(\langle x, y \rangle) = x$

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IF $\mathcal{C}$ IS A CATEGORY W/ A TERMINAL OBJECT 1 THEN A SUBJECT CLASSIFIER FOR $\mathcal{C}$ IS A $\mathcal{C}$ -OBJECT Ω TOGETHER W/ A $\mathcal{C}$ -ARROW $\text{true}: 1 \rightarrow \Omega$ THAT SATISFIES THE:

Ω -AXIOM FOR EACH MONIC $f: a \rightarrow d$ THERE IS EXACTLY ONE $\mathcal{C}$ -ARROW $\chi_f: d \rightarrow \Omega$ S.T.

$$\begin{array}{ccc} a & \xrightarrow{f} & d \\ ! \downarrow & & \downarrow \chi_f \\ 1 & \xrightarrow{\text{true}} & \Omega \end{array}$$

IS A PULLBACK SQUARE

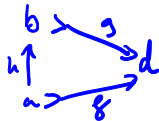
IN SET, THINK OF f AS AN INCLUSION Ω AS $\{0, 1\}$, true AS PICKING $1 \in \Omega$ AND χ_f AS A CHARACTERISTIC FUNC

INTENSIONAL CLASSES AND INTUITIONISTIC TOPOI

- TOPOI ARE MANY AND VARIED (FINITE SETS, SHEAVES, ETC.).
- THE CORE POINT IS THAT WE GET TWO KINDS OF TRUTH ABOUT SUBSETS FROM THE SUBOBJECT CLASSIFIER

1. EXTERNAL: WE LOOK AT $\text{Sub}(d) = \{[f]_\simeq \mid f: a \rightarrow d\}$
AS A LATTICE w/ RELATIONS $\subseteq, \cap, \cup$ DEFINED
USING COMMUTING SYSTEMS OF ARROWS. e.g. $f \leq g \iff \exists h$

2. INTERNAL: WE LOOK AT THE
EXPONENTIAL OBJECT Ω^d IN THE TOPOS



§3 LAWVERE'S VARIABLE SETS

- WE'LL SEE THAT WE CAN VIEW INTENSIONAL CLASSES THROUGH THE LENS OF PARTICULAR TOPOI

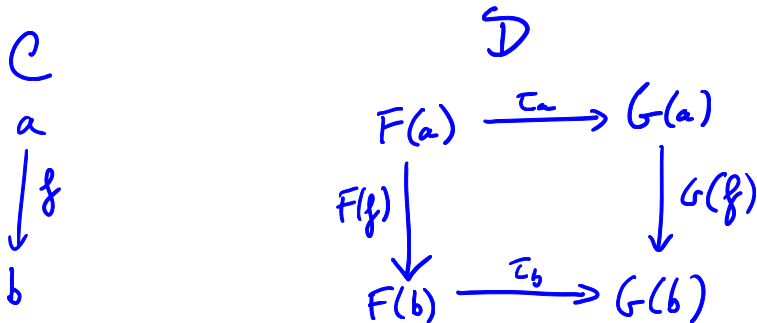
A FUNCTOR BETWEEN CATEGORIES $\mathcal{C}$ AND $\mathcal{D}$

IS A MAP THAT ASSIGNS

- (i) TO EACH $\mathcal{C}$ -OBJECT c a $\mathcal{D}$ -OBJECT $F(c)$.
- (ii) TO EACH $\mathcal{C}$ -ARROW $g: a \rightarrow b$ a $\mathcal{D}$ -ARROW $F(g)$ S.T.
 - (a) $F(\text{Id}_a) = \text{Id}_{F(a)}$
 - (b) $F(g \circ f) = F(g) \circ F(f)$

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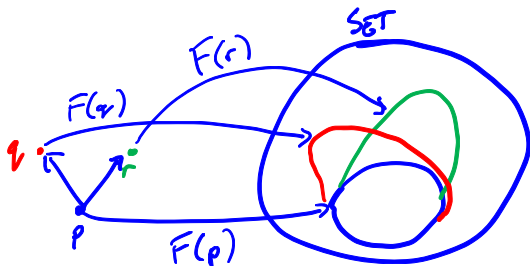
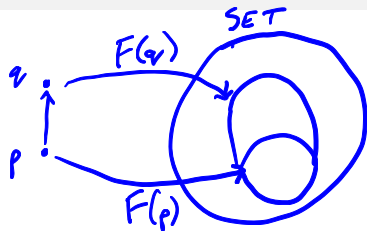
- A NATURAL TRANSFORMATION BETWEEN $F: \mathcal{C} \rightarrow \mathcal{D}$ AND $G: \mathcal{C} \rightarrow \mathcal{D}$ IS AN ASSIGNMENT τ THAT PROVIDES, FOR EACH $\mathcal{C}$ -OBJECT a , a $\mathcal{D}$ -ARROW $\tau_a: F(a) \rightarrow G(a)$, SUCH THAT FOR ANY $\mathcal{C}$ -ARROW $f: a \rightarrow b$



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- THE FUNCTOR CATEGORY $\mathcal{D}^{\mathcal{C}}$ HAS
 - AS (i) OBJECTS FUNCTORS $F: \mathcal{C} \rightarrow \mathcal{D}$
 - (ii) ARROWS NATURAL TRANSFORMATIONS
 $\tau: F \rightarrow G$
- IN DEVELOPING THESE IDEAS LAWVERE DISCOVERED THAT:
 1. YOU CAN THINK OF A PARTIAL-ORDER $\mathbb{P}$ AS A CATEGORY $\mathcal{P}$: $p \rightarrow q$ IFF $p \leq q$
 2. THE FUNCTOR CATEGORY $\text{SET}^{\mathbb{P}}$ REPRESENTS VARIABLE SETS THROUGH "TIME."

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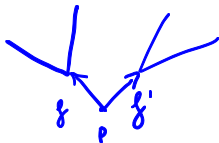
INTENSIONAL CLASSES AND INTUITIONISTIC TOPOI

FACT - Set^C IS A TOPOS FOR ANY SMALL C

- IN THE CASE OF Set^P THE SUBJECT CLASSIFIER IS A BIT FIDDLY

- $\Omega(p) =$ THE SET OF p -SIEVES.

- A p -SIEVE IS A SUBSET S OF $\{f \mid \exists g, f: p \rightarrow \tau \text{ in } P\}$
S.T. IF $f \in S$ THEN $g \circ f \in S$



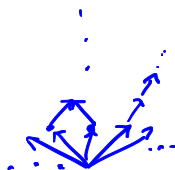
§4 APPLICATIONS TO POTENTIALISM.

- WE THINK OF $\mathbb{P}$ AS THE PARTIAL ORDER ON (A SET-SIZED PIECE OF) THE WORLDS IN A POTENTIALIST SYSTEM

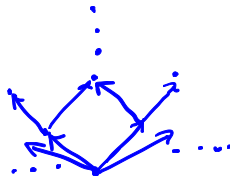
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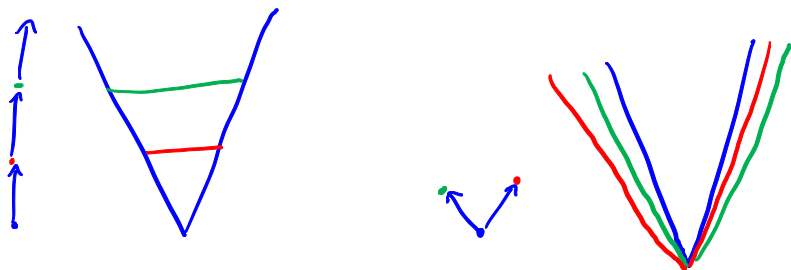


CTM



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~ AND THE POINTS w, p AS MOMENTS OF TIME IN THE POTENTIALIST SYSTEM



INTENSIONAL CLASSES AND INTUITIONISTIC TOPOI

- KEY POINT: THE STRUCTURE OF $\mathbb{P}$ IS INTIMATELY RELATED TO THE STRUCTURE OF THE SUBOBJECT CLASSIFIER
- FACT (JANUARY, 2000) $\mathbb{P}$ IS INFINITE AND LINEARLY ORDERED IFF THE LOGIC OF THE $\text{SUB}(d)$ -ALGEBRA IS INTUITIONISTIC LOGIC WITH $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$ ADDED.

INTENSIONAL CLASSES AND INTUITIONISTIC TOPOI

THIS MEANS:

1. THE LOGICAL PROPERTIES OF INTENSIONAL CLASSES (IN A ROUGH SENSE) ARE INTUITIONISTIC.
2. BOTH THE FORCING POTENTIALIST AND CTM-POTENTIALIST LOSE:

$$(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$$

§5 NEIL'S LETTER TO SANTA.

Dear Santa, This year, I would like:

1. A FULL UNDERSTANDING OF HOW THE PARTICULAR POTENTIALIST $\mathcal{P}$ RELATE TO THE PROPERTIES OF THE S.O. CLASSIFIER.

└─ FORCING POTENTIALISM HAS INCOMPATIBILITY
CTM POTENTIALISM DOES NOT.
└─ BOTH CAN HAVE LEAST ELEMENTS.

INTENSIONAL CLASSES AND INTUITIONISTIC TOPOI

2. A COMPARISON (AND BETTER UNDERSTANDING OF) THE INTERNAL AND EXTERNAL LOGICS.
3. A WAY OF HANDLING PROPER-CLASS-SIZED $\mathbb{P}$.
4. A WAY OF DEALING WITH CLASSES LOSING ELEMENTS.
5. AN ACCOUNT OF HOW THIS LINKS W/ FORCING ($\text{SET}^{\mathbb{P}_{\text{OP}}}$).

CONCLUSIONS

- REPRESENTING INTENSIONAL CLASSES
THIS WAY INDICATES THAT THEIR LOGIC
IS NON-CLASSICAL.
- THERE'S A BUNCH OF (QUASI)OPEN QUESTIONS
- THERE IS A PHILOSOPHICAL CHALLENGE OF
HOW THE POTENTIALIST SHOULD REACT.
 - └ ACCEPT?
 - └ REFINED ACCOUNT?

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THANKS

Thanks for listening, I look forward to the comments!

Hugely grateful to:

VolkswagenStiftung, Vera Flocke, Asaf Karagila, Gabe Goldberg, Mamuka Jibladze, Zhen Lin, MathOverflow, Toby Meadows, Colin McLarty, Chris Scambler, Qiaochu Yuan